Subset algebra lifting

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Starting point

Balas, Pulleyblank, Barahona, others (pre 1990).

A polyhedron $P \subseteq R^n$ can be the projection of a simpler polyhedron $Q \subseteq R^N$ ($N > n$)

More precisely:

There exist polyhedra $P \subseteq R^n$, such that

- $P$ has exponentially (in $n$) many facets, and
- $P$ is the projection of $Q \subseteq R^N$, where
- $N$ is polynomial in $n$, and $Q$ has polynomially many facets.
Given $\mathcal{F} = \{ x \in \{0, 1\}^n : Ax \geq b \}$

**Question:** Given $x^* \in \mathbb{R}^n_+$, is $x^* \in \text{conv}(\mathcal{F})$?

**Idea:** Let $N \gg n$.

Consider a function (a “lifting”) that maps each

$$v \in \{0, 1\}^n$$

into

$$\hat{z} = \hat{z}(v) \in \{0, 1\}^N$$

with $\hat{z}_i = v_i$, $1 \leq i \leq n$.

Let $\hat{\mathcal{F}}$ be the image of $\mathcal{F}$ under this operator.

**Question:** Can we find $y^* \in \text{conv}(\hat{\mathcal{F}})$, such that $y^*_i = x^*_i$, $1 \leq i \leq n$?
Concrete Idea

\( v \in \{0, 1\}^n \) mapped into \( \hat{v} \in \{0, 1\}^{2^n} \), where

(i) the entries of \( \hat{v} \) are indexed by subsets of \( \{1, 2, \ldots, n\} \), and

(ii) For \( S \subseteq \{1, \ldots, n\} \), \( \hat{v}_S = 1 \) iff \( v_j = 1 \) for all \( j \in S \).

**Example:** \( v = (1, 1, 1, 0)^T \) mapped to:

\[
\begin{align*}
\hat{v}_\emptyset &= 1, \\
\hat{v}_1 &= 1, \\
\hat{v}_2 &= 1, \\
\hat{v}_3 &= 1, \\
\hat{v}_4 &= 0, \\
\hat{v}_{\{1,2\}} &= \hat{v}_{\{1,3\}} = \hat{v}_{\{2,3\}} = 1, \\
\hat{v}_{\{1,4\}} &= \hat{v}_{\{2,4\}} = \hat{v}_{\{3,4\}} = 0, \\
\hat{v}_{\{1,2,3\}} &= 1, \\
\hat{v}_{\{1,2,4\}} &= \hat{v}_{\{1,3,4\}} = \hat{v}_{\{2,3,4\}} = 0, \\
\hat{v}_{\{1,2,3,4\}} &= 0.
\end{align*}
\]
**Definitions:** The subset *lattice* of \( \{1, 2, \cdots, n\} \) is the set of subsets of \( \{1, 2, \cdots, n\} \), ordered by inclusion.

Consider the \( 2^n \times 2^n \) matrix \( Z \), with a column \( z^p \) for each \( p \subseteq \{1, 2, \cdots, n\} \) such that

\[
z^p_q = \begin{cases} 1 & \text{if } q \subseteq p \\ 0 & \text{otherwise} \end{cases}
\]  

(1)

Note:

- The lifting maps \( v \in \{0, 1\}^n \) to \( z^{\text{supp}(v)} \).

- \( Z \) is upper-triangular, with main diagonal = \( \emptyset \)-row = 1,

- so \( Z \) is invertible. Its inverse is called the *Möbius* matrix of the lattice.

What can we say about

\[
\text{conv}\{z^p : p \in \hat{F}\}
\]

or, better, \( \text{cone}\{z^p : p \in \hat{F}\} \)?
Let $M = Z^{-1}$

If

$$y = \sum_{p \in \hat{F}} \alpha_p z^p,$$

then

$$m_p y = \alpha_p$$

($m_p = p^{th}$ row of $M$) and therefore

$cone(F)$ is defined by:

$$m_p y \geq 0, \ \forall y \in F \ (\forall p \in \hat{F}) \ \text{and}$$

$$m_p y = 0, \ \forall y \in F \ (\forall p \notin \hat{F})$$

→ An exact formulation, but in $2^n$ variables.
Take $v \in \{0, 1\}^n$. The $2^n \times 2^n$ matrix $\hat{v}\hat{v}^t$

→ Is symmetric, and its main diagonal = $\emptyset$-row.

Further, suppose $x^* \in \mathbb{R}^n$ satisfies

$$x^* = \Sigma_i \lambda_i v_i,$$

where each $v_i \in \{0, 1\}^n$, $0 \leq \lambda$, and $\Sigma_i \lambda_i = 1$.

Let $W = W(x^*) = \Sigma_i \lambda_i \hat{v}_i \hat{v}_i^t$ and $y = \Sigma_i \lambda_i \hat{v}_i$.

- $y_{\{j\}} = x^*_j$, for $1 \leq j \leq n$.

- $W$ is symmetric, $W_{\emptyset,\emptyset} = 1$, diagonal = $\emptyset$-column = $y$.

- $W \succeq 0$.

- $\forall p, q \subseteq \{1, 2, \cdots, n\}$, $W_{p,q} = y_{\{p\} \cup \{q\}}$

So we can write $W = W^y$. 
\[
x^* = \sum_i \lambda_i v_i, \quad 0 \leq \lambda \text{ and } \sum_i \lambda_i = 1.
\]

\[
y = \sum_i \lambda_i \hat{v}_i, \quad W^y = \sum_i \lambda_i \hat{v}_i \hat{v}_i^t, \quad \text{cont'd}
\]

Assume each \( v_i \in \mathcal{F} \).

**Theorem**

Suppose \( \sum_{j=1}^n \alpha_j x_j \geq \alpha_0 \forall x \in \mathcal{F} \).

Let \( p \subseteq \{1, 2, \cdots, n\} \).

Then:

\[
\sum_{j=1}^n \alpha_j W^y_{\{j\}, p} - \alpha_0 W^y_{\emptyset, p} \geq 0
\]

e.g. the \( p \)-column of \( W \) satisfies every constraint valid for \( \mathcal{F} \), homogenized.

\[ \rightarrow \text{just show that for every } i, \]

\[
\sum_{j=1}^n \alpha_j [\hat{v}_i \hat{v}_i^t]_{\{j\}, p} - \alpha_0 [\hat{v}_i \hat{v}_i^t]_{\emptyset, p} \geq 0
\]

Also holds for the \( \emptyset \)-column minus the \( u^{th} \)-column.
Lovász-Schrijver Operator

Given $\mathcal{F} = \{ x \in \{0, 1\}^n : Ax \geq b \}$

1. Form an $(n + 1) \times (n + 1)$-matrix $W$ of variables
2. **Constraint:** $W_{0,0} = 1$, $W$ symmetric, $W \succeq 0$.
3. **Constraint:** $0 \leq W_{i,j} \leq W_{0,j}$, for all $i, j$.
4. **Constraint:** The main diagonal of $W$ equals its 0-row.
5. **Constraint:** For every column $u$ of $W$,
   \[ \sum_{h=1}^{n} a_{i,h} u_h - b_i u_0 \geq 0, \forall \text{ row } i \text{ of } A \]
   and
   \[ \sum_{h=1}^{n} a_{i,h} (W_{h,0} - u_h) - b_i (1 - u_0) \geq 0, \]
   \[ \forall \text{ row } i \text{ of } A \]

Let $C = \{ x \in \mathbb{R}^n : 0 \leq x \leq 1, Ax \geq b \}$
and $N_+(C)$ = set of $x \in \mathbb{R}^n$, such that
there exists $W$ satisfying 1-5, with $W_{j,0} = x_j$, $1 \leq j \leq n$.

**Lemma.** $C \supseteq N_+(C) \supseteq N_+^2(C) \supseteq \cdots$.

**Theorem.** $N_+^n(C) = \text{conv}(\mathcal{F})$. 
\[ x^* = \sum_i \lambda_i v_i, \quad 0 \leq \lambda, \text{ each } v_i \in \mathcal{F}. \]

\[ y = \sum_i \lambda_i \hat{v}_i, \quad W^y = \sum_i \lambda_i \hat{v}_i \hat{v}_i^t, \quad \text{ again} \]

Suppose \( \alpha^T x \geq \alpha_0 \) for all \( x \in \mathcal{F} \).

Then

\[ V^{(\alpha, \alpha_0)} = \sum_i \gamma_i \hat{v}_i \hat{v}_i^T \succeq 0, \]

where

\[ \gamma_i = \lambda_i (\alpha^T v_i - \alpha_0) \geq 0. \]

The \( \emptyset \)- column of \( V^{(\alpha, \alpha_0)} \) equals \( W^y \hat{\alpha} \), where

\[ \hat{\alpha}_\emptyset = -\alpha_0 \]

\[ \hat{\alpha}_{\{j\}} = \alpha_j \quad j = 1, 2, \ldots, n \]

\[ \hat{\alpha}_p = 0, \quad \text{all other } p \]

So: \( V^{(\alpha, \alpha_0)} = W^z \), where \( z = W^y \hat{\alpha} \).
\[ \mathcal{F} = \{ x \in \{0, 1\}^n : Ax \geq b \} \]

Laurent (2001):

**Sherali-Adams level-\(k\) Operator** (*\(k = 1, 2, \cdots, n\)*) (1990)

▷ For each \(p \subseteq \{1, 2, \cdots, n\}\) with \(|p| = \min\{k + 1, n\} :\)

Form \(W[p]\), the minor of \(W\) induced by \(p\) and its subsets.

→ Require \(W[p] \succeq 0\).

▷ For each row \(a_i x \geq b_i\), and for each \(q \subseteq \{1, 2, \cdots, n\}\) with \(|q| = k\) :

Form \(V[q, i]\), the minor of \(V^{\left(a_i, b_i\right)}\) induced by \(q\) and its subsets.

→ Require \(V[q, i] \succeq 0\).

Note: we obtain the \(V[q]\) from \(W[p]\left(a_i^T, -b_i\right)\).
\[ F = \{ x \in \{0, 1\}^n : Ax \geq b \} \]

Laurent (2001):

Lasserre level-\(k\) Operator \((k = 1, 2, \ldots, n)\) (2001)

\(\triangleright\) Let \(W\{k\} = \text{minor of } W\) induced by

\[ \{p \subseteq \{1, 2, \ldots, n\} : |p| \leq \min\{k + 1, n\}\} \]

\(\rightarrow\) Require \(W\{k\} \succeq 0\).

\(\triangleright\) For each row \(a_i x \geq b_i\):

Form \(V\{i\}\), the minor of \(V^{(a_i, b_i)}\) induced by

\[ \{p \subseteq \{1, 2, \ldots, n\} : |p| \leq k\} \]

\(\rightarrow\) Require \(V\{i\} \succeq 0\).
Operator Comparison

(a) $k$-iterate convexification
   (Balas, Ceria, Cornuejols) $C^k$

(b) $k$-iterate Lovász-Schrijver $N_+^k$, $N^k$

(c) level-$k$ Sherali-Adams $S^k$

(d) level-$k$ Lasserre $L^k$

$S^k$ stronger than $N^k$ stronger than $C^k$

$L^k$ stronger than both $N_+^k$ and $S^k$

Bound on max. rank = $n$ for $N_+$ is tight:
Lovász-Schrijver revisited

\( v \in \{0, 1\}^n \) lifted to \( \hat{v} \in \{0, 1\}^{2^n} \), where

(i) the entries of \( \hat{v} \) are indexed by subsets of \( \{1, 2, \ldots, n\} \), and

(ii) For \( S \subseteq \{1, \ldots, n\} \), \( \hat{v}_S = 1 \) iff \( v_j = 1 \) for all \( j \in S \).

→ this approach makes statements about sets of variables that simultaneously equal 1

How about more complex logical statements?
Subset algebra lifting

For $1 \leq j \leq n$, let

$$Y_j = \{ z \in \{0, 1\}^n : z_j = 1 \}$$

$$N_j = \{ z \in \{0, 1\}^n : z_j = 0 \}$$

Let $\mathcal{A}$ denote the set of all set-theoretic expressions involving the $Y_j$, the $N_j$, and $\emptyset$.

Note:

(i) $\mathcal{A}$ is isomorphic to the set of subsets of $\{0, 1\}^n$.

(ii) $|\mathcal{A}| = 2^{2^n}$

(iii) $\mathcal{A}$ is a lattice under $\supseteq$

(iv) $\mathcal{A}$ contains an isomorphic copy of the lattice of subsets of $\{1, 2, \cdots, n\}$.

Lift $v \in \{0, 1\}^n$ to $\tilde{v} \in \{0, 1\}^\mathcal{A}$

where for each $S \subseteq \{0, 1\}^n$, $\tilde{v}_S = 1$ iff $v \in S$. 
Example

$v = (1, 1, 1, 0, 0) \in \{0, 1\}^5$ is lifted to

$\tilde{v} \in \{0, 1\}^{2^{64}}$ which satisfies

$\tilde{v}[(Y_1 \cap Y_2) \cup Y_5] = 1$

$\tilde{v}[Y_3 \cap Y_4] = 0$

$\tilde{v}[Y_3 \cap (Y_4 \cup N_5)] = 1$

$\ldots$

$\tilde{v}[S] = 1$ iff $(1, 1, 1, 0, 0) \in S$

Note: if $v \in \mathcal{F}$ then $\tilde{v}[\mathcal{F}] = 1$.

→ Family of algorithms that generalize Lovász-Schrijver, Sherali-Adams, Lasserre
Generic Algorithm

1. Form a family of set-theoretic indices $\mathcal{V}$. Among them, for $1 \leq j \leq n$,
   
   $Y_j$, to represent \( \{x \in \{0, 1\}^n : x_j = 1\} \)
   
   $N_j$, to represent \( \{x \in \{0, 1\}^n : x_j = 0\} \)
   
   Also $\emptyset$, $\mathcal{F}$ (and others).

2. Impose all constraints known to be valid for $\mathcal{F}$: e.g.
   
   $x_1 + 4x_2 \geq 3$ valid $\rightarrow X[Y_1] + 4X[Y_2] - 3 \geq 0$
   
   Also set theoretic constraints, e.g.
   
   $X[N_5] \geq X[Y_2 \cap N_5]$

3. Form a matrix $U \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ of variables
   
   • $U$ symmetric, main diagonal = $\mathcal{F}$-row = $X$
   
   • For $p, q \in \mathcal{V}$,
     
     $U_{p,q} = X[p \cap q]$ if $p \cap q \in \mathcal{V}$
     
     $U_{p,q} =$ a new variable, otherwise

   • Each column of $U$ satisfies all constraints,
   
   • $U \succeq 0$ (optional)

How do we algorithmically choose small $\mathcal{V}$?
Obstructions

→ An expression $\omega \in \mathcal{A}$ is an obstruction if

$$v \in \mathcal{F} \text{ implies } \bar{v}[\omega] = 0$$

(simply means: $v$ does not satisfy $\omega$)

Example: given

$$x_1 + 5x_2 + x_3 + x_4 + x_5 - 2x_6 \geq 2$$

then

$$N_1 \cap N_2 \cap Y_6$$

is an obstruction

- it is \textit{minimal}
- it is $3 -$ \textit{small}
  - at most 2 or at least 4 $Y_j$
  - at most 2 or at least 4 $N_j$

→ Set covering, set partitioning, set packing:
   all minimal obstructions are $k$-small, $k \leq 2$. 
Walls

→ A wall is an intersection (conjunction) of obstructions

**Example:**

\[ x_1 + 5x_2 + x_3 + x_4 + x_5 - 2x_6 \geq 2 \rightarrow N_1 \cap N_2 \cap Y_6 \]

\[ -x_2 + 2x_3 + x_4 + x_6 \leq 3 \rightarrow N_2 \cap Y_3 \cap Y_4 \cap Y_6 \]

\[ x_1 + x_2 + x_3 - x_4 \geq 1 \rightarrow N_1 \cap N_2 \cap N_3 \cap Y_4 \]

→ \( \omega = N_1 \cap N_2 \cap Y_4 \cap Y_6 \) is the derived **wall**

→ \( Y_1 \cap Y_2 \cap Y_4 \cap Y_6 \) is a **negation** of \( \omega \) of **order 2**

so is \( N_1 \cap N_2 \cap N_4 \cap N_6 \)

**also:**

\[ \omega^{>2} = \bigcup_{t>2} \{ \omega' : \omega' \text{ is a negation of order } t \text{ of } \omega \} \]

In general, the \( w^{>r} \) expressions are unions (disjunctions) of exponentially many intersections.
Constraints

“Box” constraints:

\[ 0 \leq X, \quad X[\mathcal{F}] = 1, \quad X[p] - X[\mathcal{F}] \leq 0 \]

Also, say:

\[ \omega = N_1 \cap N_2 \cap Y_4 \cap Y_6. \]

Then e.g.

\[ X[N_1] - X[\omega] \geq 0. \]

Also,

\[ X[Y_1] + X[Y_2] + X[N_4] + X[N_6] - 2X[\omega^{>1}] \geq 0. \]

Finally,

\[ X[\omega] + X[Y_1 \cap N_2 \cap Y_4 \cap Y_6] + X[N_1 \cap Y_2 \cap Y_4 \cap Y_6] + \\
+ X[N_1 \cap N_2 \cap N_4 \cap Y_6] + X[N_1 \cap N_2 \cap Y_4 \cap N_6] + \\
+ X[\omega^{>1}] = 1 \]

→ Implications for “matrix of variables”
Critical part of algorithm - excerpt

\( \mathcal{V} = \) family of set-theoretic variables

**Step 3.** Form a matrix \( U \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}} \) of variables

- \( U \) symmetric, main diagonal = \( \mathcal{F} \)-row = \( X \)
- For \( p, q \in \mathcal{V} \),
  \[
  U_{p,q} = X[p \cap q] \text{ if } p \cap q \in \mathcal{V} \\
  U_{p,q} = \text{a new variable, otherwise}
  \]
- Each column of \( U \) satisfies all constraints known to be valid.

\[
\begin{align*}
\star \text{ Say } \omega &= N_1 \cap N_2 \cap Y_4 \cap Y_6 \text{ is a wall.} \\
\text{Then we impose} \\
X[\omega] + X[Y_1 \cap N_2 \cap Y_4 \cap Y_6] + X[N_1 \cap Y_2 \cap Y_4 \cap Y_6] + \\
+ X[N_1 \cap N_2 \cap N_4 \cap Y_6] + X[N_1 \cap N_2 \cap Y_4 \cap N_6] + \\
+ X[\omega^{>1}] - X[\mathcal{F}] & = 0
\end{align*}
\]

→ The columns of \( U \) satisfy this equation

\star \text{ Say } p = N_1 \cap N_2, q = N_3 \cap N_4 \text{ and } N_1 \cap N_2 \cap N_3 \cap N_4 \\
\text{is an obstruction. Then} \\
U_{p,q} = 0
\]

\text{can be enforced.}
Set covering problems

\[
\min \quad c^T x \\
\text{s.t.} \quad x \in \mathcal{F}
\]

\[\mathcal{F} = \{ x \in \{0, 1\}^n : Ax \geq 1 \}, \quad A \text{ a } 0-1\text{-matrix.}\]

→ All faces \( \alpha^T x \geq \alpha_0 \) satisfy \( \alpha \geq 0 \)
→ can assume all coefficients are integral

Balas and Ng (1989):
all facets with \( \alpha_j \in \{0, 1, 2\}, j = 0, 1, \cdots, n.\)

→ There exist Gomory rank-2 valid inequalities
\[\alpha^T x \geq \alpha_0\]
where some \( \alpha_j = 3.\)
Circulant matrices

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad \sum_{j \neq h} x_j \geq 1, \quad \text{for each } h \\
& \quad x_j = 0 \text{ or } 1, \quad \text{all } j \\
\end{align*}
\]

\[
\rightarrow \sum_j x_j \geq 2 \quad \text{is valid.} \quad (2)
\]

Theorem

The combination of the Lovász-Schrijver \( N_+ \) operator and the Sherali-Adams operator requires at least rank \( n - 2 \) to guarantee (2).

\[\Rightarrow\] exponential work
\[ \mathcal{F} = \{ x \in \{0, 1\}^n : Ax \geq 1 \}, \text{ } A \text{ a 0-1 matrix.} \]

**Definition:**
The pitch of \( \alpha^T x \geq \beta \) is the smallest integer \( \pi \), such that the sum of the \( \pi \) smallest positive \( \alpha_j \) is at least \( \beta \).

e.g. \( 2x_1 + 3x_2 + 4x_3 + 4x_4 \geq 6 \) has pitch 3.

**Theorem**

Let \( k \geq 1 \), and

\[
\begin{align*}
P_k &= \{ x \in \mathbb{R}^n : 0 \leq x \leq 1, \text{ and } \text{x satisfies all valid inequalities with pitch } \leq k \}.
\end{align*}
\]

(includes \( Ax \geq 1 \))

Then: there is a polyhedron

(a) given by a formulation with polynomially many rows and columns,

(b) whose projection \( W_k \) to \( \mathbb{R}^n \) satisfies:

\[ \text{conv}(\mathcal{F}) \subseteq W_k \subseteq P_k. \]

\[ \rightarrow \text{ } P_k \text{ satisfies all inequalities with coefficients in } \{0, 1, \cdots, k\}. \]

\[ \rightarrow \text{ There are examples of set-covering problems with } \text{ exponentially } \text{ many facets with all coefficients in } \{0, 1, 2\}. \]
Additional Results

Cook and Dash, 2001 (Goemans and Tuncel, 2000):

\[ \mathcal{F} = \left\{ x \in \{0, 1\}^n : x_1 + x_2 + \cdots + x_n \geq \frac{1}{2} \right\} \]

the \( N_+ \)-rank of

\[ x_1 + x_2 + \cdots + x_n \geq 1 \]

is \( n \).

→ the subset algebra lifting algorithm proves it in polynomial-time (rank 2).

Cook, Chvátal, Hartmann (1985):

\[ \mathcal{F} = \left\{ x \in \{0, 1\}^n : \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geq \frac{1}{2}, \forall J \right\} \]

Hence, \( \mathcal{F} = \emptyset \).

The Chvátal rank is \( n \) (Cornuéjols and Li, 2001: the mixed-integer rank).

→ the subset algebra lifting algorithm proves \( \mathcal{F} = \emptyset \) in polynomial-time (rank 1).