Matrix Cube Theorems and Tight Tractable Approximations of Semi-Infinite LMIs

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1. Semi-infinite LMIs with structured norm-bounded uncertainty
   - Motivation and examples
   - Complexity status
   - Approximations

2. Approximating semi-infinite LMIs with structured norm-bounded uncertainty
Semi-infinite LMIs

A semi-infinite LMI is an infinite system of LMIs of the form

\[ A_0 + \sum_{i=1}^{n} x_i A_i \geq 0 \quad \forall [A_0, \ldots, A_n] \in \mathcal{U} \quad \text{(S)} \]

where

- \( x \) is the design vector
- \( \mathcal{U} \) is a set in the space \((S^m)^{n+1}\) of \((n+1)\)-tuples of \(m \times m\) symmetric matrices \(A_0, \ldots, A_n\).

In applications, \( \mathcal{U} \) usually arises in the form

\[ \mathcal{U} = \mathcal{U}_\rho = \{ [A_0, \ldots, A_n] = [A_0^*, \ldots, A_n^*] \]
\[ + \sum_{k=1}^{K} \delta_k [A_0^k, \ldots, A_n^k] : \delta \in \rho \Delta \}, \quad \text{(*)} \]

where

- \([A_0^*, \ldots, A_n^*]\) is the “nominal data”,
- \([A_0^k, \ldots, A_n^k]\) and \(\delta_k\) are the directions and the magnitudes of “basic perturbations”,
- \(\Delta \subset \mathbb{R}^K\) is the set of “basic perturbations of magnitude \(\leq 1\), which is a convex compact symmetric w.r.t. the origin,
- \(\rho \geq 0\) is the “uncertainty level”.

\[ A_0 + \sum_{i=1}^{n} x_i A_i \geq 0 \quad \forall [A_0, ..., A_n] \in \mathcal{U} \quad (S) \]

\[
\mathcal{U} = \{ [A_0, ..., A_n] = [A_0^*, ..., A_n^*] \\
+ \sum_{k=1}^{K} \delta_k [A_0^k, ..., A_n^k] : \delta \in \rho \Delta \} \quad (*)
\]

\[ \text{In the case of (\*)}, \ (S) \ \text{reads} \]

\[ \mathcal{A}_0(x) + \rho \sum_{k=1}^{K} \delta_k \mathcal{A}_k(x) \geq 0 \quad \forall \delta \in \Delta \]

\[ \mathcal{A}_k(x) = A_0^k + \sum_{i=1}^{n} x_i A_i^k, \ k = 0, 1, ..., K. \]

\[ \text{Sources of semi-infinite LMIs:} \]

\[ \bullet \text{Robust Counterparts of uncertain LMIs with affine data uncertainty} \]

\[ \bullet \text{Robust Control} \]

\[ \bullet \text{Some problems of maximizing convex functions over convex sets} \]
**Example:** Consider uncertain Lyapunov LMI

$$(A_* + \rho \Xi)^T X + X(A_* + \rho \Xi) \preceq -I$$

$$[A, \Xi \in \mathbb{R}^{m \times m}, X \in \mathbb{S}^m]$$  \hspace{1cm} (L)

with perturbation $\Xi$ running through a given compact set $\Delta$.

* The Robust Counterpart of (L) is the semi-infinite LMI

$$[-I - A_*^T X - XA_*] - \rho [\Xi^T X + X\Xi] \succeq 0 \quad \forall \Xi \in \Delta \quad (R)$$

* (R) is of direct interest for Control: solutions $X \succ 0$ to (R) are exactly the Lyapunov certificates for the stability of the uncertain time-varying dynamical system

$$\dot{z}(t) = A(t)z(t), \quad A(t) \in A_* + \rho \Delta$$

* (R) is related to the problem of maximizing a convex quadratic form over the unit cube, which is a NP-complete combinatorial problem.

Given $G \succ 0$, let us choose $A_*$ and $\Delta$ as

$$-I - A_*^T - A_* = G^{-1}; \quad \Delta = \{\Xi : |\Xi|_{ij} \leq 1/2\}.$$

In this case $X = I$ is feasible for (R) iff

$$G^{-1} \succeq \rho B \quad \forall (B \in \mathbb{S}^n : |B_{ij}| \leq 1)$$
\[ G^{-1} \succeq \rho B \quad \forall (B \in S^n : |B_{ij}| \leq 1) \]
\[ \Downarrow \]
\[ \xi^T G^{-1} \xi \geq \rho \max_{B=B^T : |B_{ij}| \leq 1} \xi^T B \xi \quad \forall \xi \]
\[ \Downarrow \]
\[ \xi^T G^{-1} \xi \geq \rho \|\xi\|^2_1 \quad \forall \xi \]
\[ \Downarrow \]
\[ \eta^T G \eta \leq \rho^{-1} \|\eta\|^2_\infty \quad \forall \eta \]
\[ \Downarrow \]
\[ \max_{\eta : \|\eta\|_\infty \leq 1} \eta^T G \eta \leq \rho^{-1} \]  

◆ Thus, checking whether the maximum of a given convex quadratic form over the unit cube is \( \leq \rho^{-1} \) is equivalent to checking whether \( X = I \) is a solution to a specific semi-infinite LMI — the Robust Counterpart

\[-I - A^*_X X - X A_* - \rho [\Xi^T X + X \Xi] \succeq 0 \quad \forall \Xi \in \Delta\]

of the uncertain Lyapunov LMI

\[ [A + \rho \Xi]^T X + X [A + \rho \Xi] \preceq -I \]

affected by simple-looking interval uncertainty:

\[ \Xi \in \Delta = \{ \Xi : |\Xi_{ij}| \leq 1 \}. \]
Good news: semi-infinite LMIs are important

Bad news: Already simple-looking semi-infinite LMIs, like the Robust Counterpart of the uncertain Lyapunov LMI with interval uncertainty

\[ A(X) + \rho[\Xi^T X + X \Xi] \succeq 0 \]
\[ \forall(\Xi : |\Xi_{ij}| \leq D_{ij}, i, j = 1, ..., m) \]

are NP-hard.

Conclusion: When handling intractable semi-infinite LMIs, a natural course of actions is to look for their tight tractable approximations.
Definition. We say that an LMI
\[ S_\rho(x, u) \succeq 0 \quad (A[\rho]) \]
is an approximation of the semi-infinite LMI
\[ A_0(x) + \rho \sum_{k=1}^{K} \delta_k A_k(x) \succeq 0 \quad \forall \delta \in \Delta \quad (L[\rho]) \]
if the projection \( \mathcal{Y}[\rho] \) of the solution set of \((A[\rho])\) on the space of \( x \)-variables is contained in the solution set \( \mathcal{X}[\rho] \) of \((L[\rho])\).

Approximation is called tight within factor \( \theta \geq 1 \), if
\[ \mathcal{X}[\theta \rho] \subset \mathcal{Y}[\rho] \subset \mathcal{X}[\rho], \]
or, equivalently:

(i) whenever \( x \) can be extended to a feasible solution of the approximation, \( x \) is feasible for the semi-infinite LMI of interest at the uncertainty level \( \rho \);

(ii) whenever \( x \) cannot be extended to a feasible solution of approximation, \( x \) is not feasible for the LMI of interest with increased by factor \( \theta \) uncertainty level.
\[
\mathcal{A}_0(x) + \rho \sum_{k=1}^{K} \delta_k \mathcal{A}_k(x) \succeq 0 \quad \forall \delta \in \Delta
\]

Possibilities to build tight, within moderate factors, approximations of semi-infinite LMIs depend on the structure of the LMI and the uncertainty set.

A “good case” here is given by structured norm-bounded perturbations:

\[
\mathcal{A}(x, \delta) \equiv \mathcal{A}_0(x) + \rho \sum_{k=1}^{K} \left[ L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k \right]
\]

\[\Delta = \text{Diag} \{ \Delta_1, \ldots, \Delta_K \} \in \Delta\]

where

- The \( m \times m \) matrix \( \mathcal{A}_0(x) \) is symmetric and affine in \( x \);
- The matrices \( L_k, R_k(x) \) are \( d_k \times m \), and \( R_k(x) \) are affine in \( x \);
- Perturbations \( \Delta = \text{Diag}\{ \Delta_1, \ldots, \Delta_K \} \) are block-diagonal matrices with \( K \) diagonal blocks of the sizes \( d_1, \ldots, d_K \);
- \( \Delta \) is comprised of all \( \Delta = \text{Diag}\{ \Delta_1, \ldots, \Delta_K \} \) such that

\[\| \Delta_k \| \leq 1, \quad k = 1, \ldots, K; \quad \Delta_k = \delta_k I_{d_k}, \quad k \in I_s.\]
\[ A_0(x) + \rho \sum_{k=1}^{K} \left[ L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k \right] \succeq 0 \]
\[ \forall \left( \Delta = \{ \Delta_k \} : \| \Delta_k \| \leq 1, k \leq K \right. \]
\[ \left. \Delta_k = \delta_k I_{d_k}, k \in \mathcal{I}_s \right) \]

- **Example:** Semi-infinite LMI with “interval uncertainty”

\[ A_0(x) + \rho \sum_{k=1}^{K} \delta_k A_k(x) \succeq 0 \quad \forall (\delta \in \mathbb{R} : \| \delta \|_\infty \leq 1) \]

can be rewritten equivalently as

\[ A_0(x) + \rho \sum_{k=1}^{K} [L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k] \succeq 0 \quad \forall \Delta \in \Delta \]

where \( \Delta \) corresponds to “repeated scalar perturbations” (\( \mathcal{I}_s = \{1, \ldots, K\} \)) and \( L_k, R_k(x) \) are given by

\[ A_k(x) = L_k^T R_k(x) + R_k^T(x) L_k. \]
• Special case: The semi-infinite Lyapunov LMI with interval uncertainty

\[
\begin{bmatrix}
-I - A^T_* X - X A_* & + \rho [\Xi^T X + X \Xi]
\end{bmatrix} \succeq 0
\]

\( \mathcal{A}_0(x) \)

\( \forall (\Xi : |\Xi_{ij}| \leq D_{ij}, i, j = 1, ..., m) \)

\[ \uparrow \]

\( \mathcal{A}_0(x) + \rho \sum_{i,j} \delta_{ij} D_{ij} [e_j e_i^T X + X e_i e_j^T] \succeq 0 \)

\( \forall (\delta = \{\delta_{ij}\} : |\delta_{ij}| \leq 1) \)

\[ \downarrow \]

\[
\mathcal{A}_0(X) + \rho \sum_{i,j} \left[ L_{ij} \delta_{ij} R_{ij}(X) + R_{ij}^T(X) \delta_{ij} L_{ij} \right] \succeq 0
\]

\( \forall \Delta = \text{Diag}\{\delta_{ij}\} \in \Delta \)

\[
\begin{bmatrix}
L_{ij} = D_{ij} e_j^T, R_{ij}(X) = e_i^T X
\end{bmatrix}
\]

\( d_{ij} = 1, i, j = 1, ..., m, \)

\( \mathcal{I}_s = \{(i, j) : 1 \leq i, j \leq m\} \)
Remark: In the description of structured norm-bounded uncertainty

\[
\Delta = \left\{ \text{Diag}\{\Delta_1, ..., \Delta_K\} : \|\Delta_k\| \leq 1, \; k \leq K \right\}
\]

\[
\Delta_k = \delta_k I_{d_k}, \; k \in \mathcal{I}_s
\]

1 \times 1 perturbation blocks $\Delta_k$ can be considered both as scalar ($k \in \mathcal{I}_s$) and as full ($k \not\in \mathcal{I}_s$). It is convenient to treat these blocks as full. Thus, from now on

\[
k \in \mathcal{I}_s \Rightarrow d_k \geq 2.
\]

In particular, from now on we treat the Lyapunov LMI with interval uncertainty as the semi-infinite LMI

\[
\mathcal{A}_0(X) + \rho \sum_{i,j} \left[ L_{ij} \delta_{ij} R_{ij}(X) + R_{ij}^T(X) \delta_{ij} L_{ij} \right] \succeq 0
\]

\[
\forall \Delta = \text{Diag}\{\delta_{ij}\} \in \Delta
\]

\[
\begin{bmatrix}
L_{ij} = D_{ij} e_j^T, \; R_{ij}(X) = e_i^T X \\
d_{ij} = 1, \; i, j = 1, ..., m, \\
\mathcal{I}_s = \emptyset
\end{bmatrix}
\]

with full $1 \times 1$ perturbation blocks.
Matrix Cube Theorem [Ben-Tal, Nemirovski, Roos, 2001]. Consider a semi-infinite LMI with structured norm-bounded uncertainty

\[
\mathcal{A}_0(x) + \rho \sum_{k=1}^{K} [L_k \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k] \succeq 0
\]

\[
\forall \Delta = \{\Delta_k\} : \|\Delta_k\| \leq 1, \ k \leq K, \ \Delta_k = \delta_k I_{d_k}, \ k \in \mathcal{I}_s
\]

(R[\rho])

The system of LMIs in variables \(x, X_k \in S^m, \lambda_k \in \mathbb{R}, k \notin \mathcal{I}_s\):

\[
X_k \succeq \pm [L_k R_k(x) + R_k^T(x) L_k], \ k \in \mathcal{I}_s
\]

\[
\begin{bmatrix}
X_k - \lambda_k L_k^T L_k & R_k^T(x) \\
R_k(x) & \lambda_k I_{d_k}
\end{bmatrix} \succeq 0, \ k \notin \mathcal{I}_s
\]

(A[\rho])

\[
\mathcal{A}_0(x) \succeq \rho \sum_{k=1}^{K} X_k
\]

is an approximation of (R[\rho]) tight within the factor

\[
\vartheta \left( \max_{k \in \mathcal{I}_s} d_k \right) = 1
\]

\[
\vartheta(\mu) \text{ is a universal function of } \mu \text{ such that }
\]

\[
\vartheta(1) = \frac{\pi}{2} = 1.57..., \ \vartheta(2) = 2, \ \vartheta(\mu) \leq \sqrt{2\pi \mu}.
\]

Besides this, in the case \(K = 1\) of a single perturbation block, (A[\rho]) is equivalent to (R[\rho]).
**Sketch of the proof.**

**A. A simple sufficient condition for the validity of the LMI**

\[ \mathcal{A}_0(x) + \rho \sum_{k=1}^{K} [L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k] \succeq 0 \]

for all perturbations \( \Delta_k \) satisfying \( \| \Delta_k \| \leq 1 \), \( \Delta_k = \delta_k I_{d_k}, k \in \mathcal{I}_s \), is the existence of matrices \( X_k \) such that

\[
\begin{align*}
(a) \quad X_k &\succeq L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k \\
&\forall (\Delta_k : \| \Delta_k \| \leq 1 \& \Delta_k = \delta_k I_{d_k}, k \in \mathcal{I}_s) \\
(b) \quad \mathcal{A}_0(x) &\succeq \rho \sum_{k=1}^{K} X_k
\end{align*}
\]

For \( k \in \mathcal{I}_s \), (a) is clearly equivalent to

\[ X_k \succeq \pm [L_k^T R_k(x) + R_k^T(x) L_k], \]

while for \( k \notin \mathcal{I}_s \), (a) is equivalent to

\[ \exists \lambda_k : \begin{bmatrix} X_k - \lambda_k L_k^T L_k & R_k^T(x) \\
R_k(x) & \lambda_k I_{d_k} \end{bmatrix} \succeq 0 \]

With these observations, our sufficient condition becomes exactly \( \mathcal{A}[^{\rho}] \); thus, \( \mathcal{A}[^{\rho}] \) is an approximation of \( \mathcal{R}[^{\rho}] \).
B. In order to bound the tightness factor of the approximation, assume that a given \( x \) cannot be extended to a feasible solution of \( (A[\rho]) \); we should prove that then \( x \) is infeasible for \( (R[\vartheta(\max_{k \in I_s} d_k)]) \).

Our assumption is equivalent to the fact that the optimal value in the semidefinite program

\[
\min_{\tau, \{X_k\}, \{\lambda_k\}} \left\{ \tau : \begin{bmatrix} X_k & \pm A_k, & k \in I_s \\
X_k - \lambda_k L_k^T L_k & R_k^T & R_k \\
R_k & \lambda_k I_{d_k} & \end{bmatrix} \preceq 0, \ k \not\in S \\
\tau I + A \succeq \rho \sum_k X_k \\
\right\}
\]

is positive. Since the problem is strictly feasible, this means that the dual problem admits a feasible solution with positive value of the objective. This reduces to

\[
\exists Z \succeq 0 : \\
\rho \left[ \sum_{k \in I_s} \|\lambda(Z^{1/2} A_k Z^{1/2})\|_1 + 2 \sum_{k \not\in I_s} \|L_k Z^{1/2}\|_2 \|R_k Z^{1/2}\|_2 \right] \\
> \text{Tr}(Z^{1/2} A Z^{1/2})
\]

where \( \lambda(B) \) is the vector of eigenvalues of \( B \in S^m \) and \( \|B\|_2 = \sqrt{\text{Tr}(BB^T)} \).
\[ \exists Z \succeq 0 : \]
\[ \rho \left[ \sum_{k \in I_s} \| \lambda(Z^{1/2}A_kZ^{1/2}) \|_1 + 2 \sum_{k \notin I_s} \| L_kZ^{1/2} \|_2 \| R_kZ^{1/2} \|_2 \right] > \text{Tr}(Z^{1/2}AZ^{1/2}) \tag{*} \]

**Lemma:** Let \( \xi \sim \mathcal{N}(0, I_m) \). Then

(i) For a matrix \( B \in \mathbb{S}^m \), one has
\[ \mathbb{E} \{ |\xi^T B \xi| \} \geq \vartheta^{-1}(\lfloor \text{Rank}(B)/2 \rfloor) \| \lambda(B) \|_1 \]

(ii) For matrices \( P, Q \in \mathbb{R}^{d \times m} \),
\[ \mathbb{E} \{ \| P \xi \|_2 \| Q \xi \|_2 \} \geq \vartheta^{-1}(1) \| P \|_2 \| Q \|_2. \]

Here \( \vartheta(\cdot) \) is as required in the MCT. Setting
\[ \mu = \max_{k \in I_s} d_k, \]
so that
\[ k \in I_s \Rightarrow \text{Rank}(Z^{1/2}A_kZ^{1/2}) \leq 2\mu, \]
we get from Lemma combined with \( (*) \) the inequality
\[ \rho \vartheta(\mu) \mathbb{E} \left\{ \sum_{k \in I_s} |\xi^T Z^{1/2}A_kZ^{1/2} \xi| \right\} \\
+ 2 \sum_{k \notin I_s} \| L_kZ^{1/2} \xi \|_2 \| R_kZ^{1/2} \xi \|_2 \} > \text{Tr}(Z^{1/2}AZ^{1/2}) \\
= \mathbb{E} \{ \xi^T Z^{1/2}AZ^{1/2} \xi \} \]
\[ \rho \vartheta(\mu) \mathbb{E}\left\{ \sum_{k \in \mathcal{I}_s} |\xi^T Z^{1/2} A_k Z^{1/2} \xi| + 2 \sum_{k \notin \mathcal{I}_s} \|L_k Z^{1/2} \xi\|_2 \|R_k Z^{1/2} \xi\|_2 \right\} > \text{Tr}(Z^{1/2} A Z^{1/2}) = \mathbb{E}\{\xi^T Z^{1/2} A Z^{1/2} \xi\} \]

It follows that there exists a realization \( \zeta \) of the random vector \( Z^{1/2} \xi \) such that

\[ \rho \vartheta(\mu) \left\{ \sum_{k \in \mathcal{I}_s} |\xi^T A_k \xi| + 2 \sum_{k \notin \mathcal{I}_s} \|L_k \xi\|_2 \|R_k \xi\|_2 \right\} > \zeta^T A \zeta \quad (\star) \]

- For \( k \in \mathcal{I}_s \), we can choose \( \Delta_k = \pm I_{d_k} \) in such a way that
  \[ |\zeta^T A_k \zeta| = |\zeta^T [L_k^T R_k + R_k^T L_k] \zeta| = \zeta^T [L_k^T \Delta_k R_k + R_k^T \Delta_k^T L_k] \zeta \]

- For \( k \notin \mathcal{I}_s \), we can choose \( \Delta_k \in \mathbb{R}^{d_k \times m}, \|\Delta_k\| = 1 \), in such a way that
  \[ 2 \|L_k \zeta\|_2 \|R_k \zeta\|_2 = \zeta^T [L_k^T \Delta_k R_k + R_k^T \Delta_k^T L_k] \zeta \]

Recalling that \( A = A_0(x) \), \( R_k = R_k(x) \), (\star) reads

\[ \zeta^T \left[ A_0(x) - \rho \vartheta(\mu) \sum_k [L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k] \right] \zeta < 0, \]

while by construction \( \|\Delta_k\| \leq 1 \) and \( \Delta_k = \delta_k I_{d_k} \) for \( k \in \mathcal{I}_s \). Thus,

\[ x \notin \mathcal{X}[\vartheta(\mu) \rho]. \]
Application, I: Maximizing convex quadratic form over the unit cube.

We have seen that if $G \succeq 0$, then

$$
\omega(G) \equiv \max_{\|\eta\|_{\infty} \leq 1} \eta^T G \eta = \frac{1}{\rho^*}, \\
\rho^* = \max \{ \rho : G^{-1} \succeq \rho A \forall (A = A^T : |A_{ij}| \leq 1) \}.
$$

We have

$$
\rho^* = \max \{ \rho : G^{-1} + \rho \sum_{i \leq j} [L_{ij}^T \Delta_{ij} R_{ij} + R_{ij}^T \Delta_{ij}^T L_{ij}] \succeq 0 \\
\forall (\Delta_{ij} \in \mathbb{R} : |\Delta_{ij}| \leq 1) \}
$$

$$
L_{ij} = \begin{cases}
    e_i^T, & i \neq j \\
    \frac{1}{\sqrt{2}} e_i, & i = j
\end{cases}, \\
R_{ij} = \begin{cases}
    e_j^T, & i \neq j \\
    \frac{1}{\sqrt{2}} e_j, & i = j
\end{cases}
$$

By the MCT, the efficiently computable quantity

$$
\bar{\rho} = \max_{\rho, \{x_{ij}\}, \{\lambda_{ij}\}} \left\{ \rho : \begin{bmatrix}
    X_{ij} - \lambda_{ij} L_{ij}^T L_{ij} & R_{ij}^T \\
    R_{ij} & \lambda_{ij}
\end{bmatrix} \succeq 0, i \leq j \\
G^{-1} \succeq \rho \sum_{i \leq j} X_{ij}
\right\}
$$

is a lower bound, tight within the factor $\vartheta(1) = \frac{\pi}{2}$, on $\rho^*$, so that $\frac{1}{\bar{\rho}}$ is an efficiently computable upper bound on $\omega(G)$. 
On a closest inspection, the efficiently computable upper bound
\[
\frac{1}{\tilde{\rho}} = \max_{\rho, X_{ij} \lambda_{ij}} \left\{ \rho : \begin{bmatrix} X_{ij} - \lambda_{ij} L_{ij}^T L_{ij} & R_{ij}^T \\ R_{ij} & \lambda_{ij} \end{bmatrix} \succeq 0, \ i \leq j \right\}
\]
on \omega(G) = \max_{\eta : \|\eta\| \leq 1} \eta^T G \eta turns out to be exactly the standard semidefinite relaxation bound
\[
\tilde{\omega}(G) = \min_{\lambda} \left\{ \sum_i \lambda_i : \text{Diag}\{\lambda\} \succeq G \right\}
\equiv \max_X \left\{ \text{Tr}(GX) : X \succeq 0, X_{ii} = 1 \right\},
\]
and we arrive at \(\frac{\pi}{2}\)-Theorem of Nesterov (1996):

For \(G \succeq 0\), \(\tilde{\omega}(G)\) is a tight, within the factor \(\frac{\pi}{2}\), upper bound on \(\omega(G)\).

which originally was proved via the random hyperplane technique of Goemans and Williamson.
Lyapunov Stability Analysis under interval uncertainty

The possibility to certify the stability of uncertain time-varying dynamical system

\[ \dot{z}(t) = A(t)z(t) \quad [A(t) \in \mathcal{U}\forall t] \]

by a Lyapunov stability certificate is equivalent to solvability of the semi-infinite LMI

\[ X \succeq I, \quad A^T X + X A \preceq -I \quad \forall A \in \mathcal{U} \quad (L) \]

Assume that uncertainty comes from structured norm-bounded perturbations:

\[ \mathcal{U} = \left\{ A = A_* + \rho \sum_{k=1}^{K} P_k^T \Delta_k^T Q_k : \|\Delta_k\| \leq 1, k \leq K \right\} \]

and that we are interested to compute the Lyapunov Stability Radius \( \rho^* \) — the supremum of those \( \rho \) for which (L) is solvable.
With structured norm-bounded uncertainty, the semi-infinite LMI Lyapunov LMI reads

\[
\begin{align*}
& A_0(X) \\
& [ -I - A_*^T X - X A_* ] \\
& + \rho \sum_{k=1}^{K} [ Q_k^T \Delta_k (P_k X) + (P_k X)^T \Delta_k^T Q_k ] \geq 0 \\
& \forall \{ \Delta_k \} : \| \Delta_k \| \leq 1, k \leq K \\
& \Delta_k = \delta_k I_{d_k}, k \in \mathcal{I}_s
\end{align*}
\]

The MCT implies a sufficient condition for solvability of (\*) and thus – an efficiently computable lower bound \( \bar{\rho} \) on \( \rho^* \):

\[
\bar{\rho} = \sup_{\rho, X, \{ X_k \}, \{ \lambda_k \}} \left\{ \begin{array}{l}
X \succeq I \\
X_k \succeq \pm [ Q_k^T P_k X + X P_k^T Q_k ], k \in \mathcal{I}_s \\
\rho : \left[ \begin{array}{cc}
X_k - \lambda_k Q_k^T Q_k & XP_k^T \\
P_k X & \lambda I_{d_k}
\end{array} \right] \succeq 0, k \not\in \mathcal{I}_s \\
A_0(X) \succeq \rho \sum_k X_k
\end{array} \right. 
\]

The bound is tight within the factor \( \vartheta \left( \max_{k \in \mathcal{I}_s} d_k \right) \):

\[
\bar{\rho} \leq \rho^* \leq \vartheta \left( \max_{k \in \mathcal{I}_s} d_k \right) \bar{\rho}.
\]

For example, in the case of interval uncertainty \( d_k = 1, k = 1, \ldots, K \), the factor becomes \( \frac{\pi}{2} \).
Many important properties of a linear time-invariant dynamical system

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) \\
y(t) &= Cz(t) + Du(t)
\end{align*}
\]  

are LMI-representable — (S) possesses the property iff certain LMI

\[
\Sigma \begin{bmatrix} A & B \\ C & D \end{bmatrix}(X) \succeq 0
\]

associated with (S) is solvable:

\[\blacklozenge\] Open-loop stability:

\[
\lim_{t \to \infty} u(t) = 0 \Rightarrow \lim_{t \to \infty} z(t) = 0
\]

\[
\Leftrightarrow \exists X : X \succeq I, A^TX +XA \preceq -I
\]

\[\blacklozenge\] Stabilizability via state feedback:

\[
\exists K : \begin{cases}
\dot{z} = Az + Bu \\
u = Kz
\end{cases} \Rightarrow \lim_{t \to \infty} z(t) = 0
\]

\[
\Leftrightarrow \exists X = [Y, Z] : Y \succeq I, YA^T + AY + ZB^T + BZ^T \preceq -I
\]
 Positive realness:

\[ z(0) = 0 \Rightarrow \int_{0}^{T} u^{T}(t)y(t)dt \geq 0 \quad \forall T \geq 0 \]

\[ \iff \quad \exists X : X \succ 0, \begin{bmatrix} A^{T}X +XA & XB - C^{T} \\ B^{T}X - C & -D^{T} - D \end{bmatrix} \preceq 0 \]

 Real boundedness:

\[ z(0) = 0 \Rightarrow \int_{0}^{T} y^{T}(t)y(t)dt \leq \int_{0}^{T} u^{T}(t)u(t)dt \quad \forall T \geq 0 \]

\[ \iff \quad \exists X : X \succ 0, \begin{bmatrix} A^{T}X +XA & XB & C^{T} \\ B^{T}X & -I & D^{T} \\ C & D & -I \end{bmatrix} \preceq 0 \]
In all outlined (and many other) cases, solvability of the semi-infinite LMI

\[ \Sigma \begin{bmatrix} A & B \\ C & D \end{bmatrix}(X) \succeq 0 \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{U} \quad (*) \]

is a sufficient condition for the property to be possessed by the uncertain time-varying system

\[ \begin{aligned}
\dot{z}(t) &= A(t)z(t) + B(t)u(t) \\
y(t) &= C(t)z(t) + D(t)u(t),
\end{aligned} \quad \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathcal{U} \ \forall t
\]

Assuming structured norm-bounded uncertainty

\[ \mathcal{U} = \left\{ \begin{bmatrix} A_* & B_* \\ C_* & D_* \end{bmatrix} + \rho \sum_{k} P_k \Delta_k Q_k^T : \|\Delta_k\| \leq 1, k \leq K \right\} \]

\[ \Delta_k = \delta_k I_{d_k}, \quad k \in \mathcal{I}_s \]

and with “well-structured \( \Sigma \)” (as it is the case in all our examples), the MCT yields an \( O(1) \sqrt{\max_{k \in \mathcal{I}_s} d_k} \)-tight computationally tractable approximation to the semi-infinite LMI \( (*) \).
Consider the analysis version of a semi-infinite LMI with structured norm-bounded uncertainty and scalar perturbation blocks

\[ \rho^* = \max \left\{ \rho : A + \rho \sum_{k=1}^{K} \delta_k A_k \succeq 0 \forall (\delta \in \mathbb{R}^K : \|\delta\|_{\infty} \leq 1) \right\} \]

When solving this problem, we lose nothing by assuming that \( A \succ 0 \). In this case, the scaling \( A \leftarrow I, A_k \leftarrow A^{-1/2} A_k A^{-1/2} \) converts (P) into the problem

\[ \frac{1}{\rho^*} = \min \left\{ \lambda : \| \sum_k \delta_k A_k \| \leq \lambda \ \forall (\delta : \|\delta\|_{\infty} \leq 1) \right\} \]

which is the problem of computing the norm of the linear mapping

\[ A(\delta) = \sum_{k=1}^{K} \delta_k A_k : (\mathbb{R}^K, \| \cdot \|_{\infty}) \to (\mathbb{S}^m, \| \cdot \|) \]

The MCT offers an efficiently computable upper bound, tight within the factor \( O(1) \sqrt{\max_k \text{ Rank}(A_k)} \), on this norm.

What about computing the norm of the same mapping regarded as a mapping from \( (\mathbb{R}^K, \| \cdot \|_p) \) into \( (\mathbb{R}^K, \| \cdot \|_{\infty}) \to (\mathbb{S}^m, \| \cdot \|) \)?
How to compute/estimate the norm of a linear mapping

$$A(\delta) = \sum_{k=1}^{K} \delta_k A_k : (\mathbb{R}^K, \| \cdot \|_p) \to (\mathbb{S}^m, \| \cdot \|)$$

The case of utmost interest here is $p = 2$, where, our question reduces to the following NP-hard problem:

*Given a $K$-dimensional ellipsoid in $\mathbb{S}^n$, centered at a positive definite matrix, what is the largest similar ellipsoid with the same center which is contained in the positive semidefinite cone?*
\[ \mathcal{A}(\delta) = \sum_{k=1}^{K} \delta_k A_k : \mathbb{R}^K \to \mathbb{S}^m. \]

By the standard arguments, the function

\[ \phi(\alpha) \equiv \ln(\|\mathcal{A}\|_1^\alpha) = \ln \left( \max \{ \|\mathcal{A}(\delta)\| : \|\delta\|_1^\alpha \leq 1 \} \right) \]

is

- convex and nonincreasing in \( \alpha \in [0, 1] \)
- Lipschitz continuous, with constant \( \ln(K) \).

By elementary arguments, it follows that

\[
\begin{align*}
\phi(\alpha) &\leq (1 - \alpha)\phi(0) + \alpha\phi(1) \\
\phi(\alpha) &\geq (1 - \alpha)\phi(0) + \alpha\phi(1) - \alpha(1 - \alpha) \ln(K)
\end{align*}
\]

\[ \Downarrow \]

\[ K^{-\frac{p-1}{p^2}} \|\mathcal{A}\|^{1-1/p}_\infty \|\mathcal{A}\|_1^{1/p} \leq \|\mathcal{A}\|_p \leq \|\mathcal{A}\|^{1-1/p}_\infty \|\mathcal{A}\|_1^{1/p} \]

The quantity \( \|\mathcal{A}\|_1 \) is easily computable:

\[ \|\mathcal{A}\|_1 = \max_k \|A_k\|, \]

while the MCT provides us with an upper bound

\[ \gamma_\infty(\mathcal{A}) = \min_{\gamma \in \{X_k \}} \left\{ \gamma : X_k \succeq \pm A_k \right\} \]

on \( \|\mathcal{A}\|_\infty \), and this bound is tight within the factor \( O(1) \sqrt{\max_k \text{Rank}(A_k)} \).
\[ \mathcal{A}(\delta) = \sum_{k=1}^{K} \delta_k A_k : \mathbb{R}^K \rightarrow \mathbb{S}^m. \]

\[ \mathbf{\Delta} \]

We arrive at the efficiently computable upper bound

\[ \gamma_p(\mathcal{A}) = \left( \max_k \| A_k \| \right)^{1-1/p} \gamma_\infty^{1/p}(\mathcal{A}) \]

on

\[ \| \mathcal{A} \|_p = \max \left\{ \| \sum_{k=1}^{K} \delta_k A_k \| : \| \delta \|_p \leq 1 \right\}, \]

and this bound is tight within the factor

\[ O(1) \left( \max_k \text{Rank}(A_k) \right)^{1/2p} K^{p-1/\left(p^2\right)}. \]

When \( p = 2 \) and the ranks of \( A_k \) are \( O(1) \), the factor becomes \( O(1)K^{1/4} \) (cf. the factor \( \sqrt{\min[K,m]} \) for the only previously known computable upper bound on \( \| \mathcal{A} \|_2 \)).
The MCT admits complex case extension: Matrix Cube Theorem, Complex Case: Consider a semi-infinite LMI

\[
\mathcal{A}_0(x) + \sum_{k=1}^{K} \left[ L_k^H \Delta_k R_k(x) + R_k^H(x) \Delta_k^H L_k \right] \geq 0
\]

\[
\forall \left\{ \Delta_k \right\} : \\
\Delta_k \in \mathbb{C}^{d_k \times d_k}, \ k \leq K \\
\| \Delta_k \| \leq 1, \ k \leq K \\
\Delta_k = \delta_k I_{d_k}, \ \delta_k \in \mathbb{R}, \ k \in \mathcal{T}_s^R \\
\Delta_k = \delta_k I_{d_k}, \ \delta_k \in \mathbb{C}, \ k \in \mathcal{T}_s^C
\]

where \( L_k, R_k(x) \in \mathbb{C}^{d_k \times m} \), \( \mathcal{A}_0(x) \) is Hermitian and \( R_k(x), \mathcal{A}_0(x) \) are affine in \( x \).

The system of LMIs in Hermitian matrix variables \( X_k, V_k \) and real variables \( \lambda_k \)

\[
X_k \succeq \pm \left[ L_p^H R_p(x) + R_p^H(x) L_p \right], \ k \in \mathcal{T}_s^R, \\
\begin{bmatrix}
X_k - V_k & L_k^H R_k(x) \\
R_k^H(x) L_k & V_k
\end{bmatrix} \succeq 0, \ k \in \mathcal{T}_s^C, \\
\begin{bmatrix}
X_k - \lambda_k L_k L_k^H R_k(x) \\
R_k(x) & \lambda_k I_{d_k}
\end{bmatrix} \succeq 0, \ k \not\in \mathcal{T}_s^R \cup \mathcal{T}_s^C
\]

\[
A - \rho \sum_{k=1}^{K} X_k \succeq 0
\]

is a tight, within the factor \( \varphi_C \left( \max_{k \in \mathcal{T}_s^R \cup \mathcal{T}_s^C} d_k \right) \), approximation of \( (\mathcal{R}[\rho]) \). Here \( \varphi_C(\mu) \leq O(1) \sqrt{\mu} \) is a universal function such that \( \varphi_C(1) = \frac{4}{\pi} \).
Corollary 1. For a Hermitian $m \times m$ matrix $G \succeq 0$, the Semidefinite Relaxation bound

$$
\bar{\omega}(G) = \min_{\lambda} \left\{ \sum \lambda_i : \text{Diag}\{\lambda\} \succeq G \right\}
$$
on the quantity

$$
\omega(G) = \max_{\eta \in \mathbb{C}^m : \|\eta\|_{\infty} \leq 1} \eta^H G \eta,
$$
and this bound is tight within the factor $\frac{4}{\pi} = 1.27...$:

$$
\omega(G) \leq \bar{\omega}(G) \leq \frac{4}{\pi} \omega(G).
$$
Corollary 2. Consider a time-varying uncertain dynamical system

$$\dot{z}(t) = A(t)z(t), \quad A(t) \in \mathcal{U} \ \forall t$$  \hspace{1cm} (S)

with complex data. In the case of structured norm-bounded uncertainty

$$\mathcal{U} = \left\{ A_* + \rho \sum_{k=1}^{K} P_k^H \Delta_k Q_k : \begin{array}{l} \Delta_k \in \mathbb{C}^{d_k \times d_k}, k \leq K \\ \|\Delta_k\| \leq 1, k \leq K \\ \Delta_k = \delta_k I_{d_k}, \delta_k \in \mathbb{R}, k \in \mathcal{I}_s^R \\ \Delta_k = \delta_k I_{d_k}, \delta_k \in \mathbb{C}, k \in \mathcal{I}_s^C \end{array} \right\}$$

the Lyapunov Stability Radius \( \rho^* \) of (S):

$$\rho^* = \sup \{ \rho : \exists X \succeq I : A^H X + X A^H \preceq -I \ \forall A \in \mathcal{U} \}$$

admits a tight, within the factor \( O(1) \sqrt{\max_{k \in \mathcal{I}_s^R \cup \mathcal{I}_s^C} d_k} \), efficiently computable lower bound. In the case of \( \mathcal{I}_s^C = \mathcal{I}_s^R = \emptyset \) (“interval uncertainty”), the bound is tight within the factor \( \frac{4}{\pi} \).