Adapting to nonstationary behavior
Examples from geophysics and cosmology

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Objectives

- Improve modeling → reduce bias

- Adapt to nonstationary behavior

- More flexible to constraints

Main tool: Gaussian mixtures
Examples

• Reflection seismology: Earth’s reflectivity

\[ \text{signal} = \text{flat} + \text{bursts} \]

• Cosmology: sources in far-IR sky maps

\[ \text{map} = \text{background} + \text{local sources} \]
Gaussian Mixtures

• Sample randomly from Gaussians

• \( F(x) = \alpha_1 F_1(x) + \cdots + \alpha_k F_k(x), \ \sum \alpha = 1 \)

• Flexible modeling, accessible inference
Convolutional Model

- \( s_t = \sum_k w_k r_{t-k} + z_t = w_t * r_t + z_t \)

\( s_t \) Seismic trace —data

\( w_t \) Seismic pulse —unknown

\( r_t \) Reflectivity —unknown

\( z_t \) Noise
Objective

- Estimate \( r_t \) (deconvolve)

- Deconvolution filter \( g_t \) \((g_0 = 1)\)

\[
g_t * s_t = s_t + \sum_{k=1}^{p} g_k s_{t-k} \approx r_t
\]
Wiener-Levinson

- Yule-Walker equations

\[
\begin{pmatrix}
1 & \hat{\rho}(1) & \cdots & \hat{\rho}(p-1) \\
\hat{\rho}(1) & 1 & \cdots & \hat{\rho}(p-2) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\rho}(p-1) & \hat{\rho}(p-2) & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\hat{\rho}(1) \\
\hat{\rho}(2) \\
\vdots \\
\hat{\rho}(p)
\end{pmatrix}
= g
\]

- Durbin-Levinson algorithm
Why not Wiener-Levinson?

- Yule-Walker equations $\rightarrow$ WL filter $g_t$

- Simple $\rightarrow$ widely used

- Reasonably robust?

  Weaken assumptions $\rightarrow$ more adaptable
WL Assumptions

- $r_t$ white (Gaussian)

- $w_t$ invertible

- $s_t$ noiseless
In Practice

- \( r_t \) not stationary
- \( r_t \) not Gaussian
- \( w_t \) not invertible
- Correlated noise
Robustify WL

- Better adapt to nonstationarities

- Model non-Gaussian predictions

- Natural generalization of WL
If $r_t$ is white

$$g_t \ast s_t = s_t + \sum_{k=1}^{p} g_k s_{t-k} = r_t$$

$$\hat{s}_t = E(s_t | S^{t-1}) = - \sum_{k=1}^{p} g_k s_{t-k}$$

$$r_t = s_t - \hat{s}_t$$

Filter $= (1, g_1, \ldots, g_p)$
Generalize

\[
WL : \quad F(s_t \mid S^{t-1}) = N \left( - \sum_{i=1}^{p} g_is_{t-i}, \sigma^2 \right)
\]

Generalize : \quad F(s_t \mid S^{t-1}) = \alpha_0 N \left( - \sum_{i=1}^{p} g_is_{t-i}, \sigma_0^2 \right) + \alpha_1 N \left( \phi_1 s_{t-1}, \sigma_1^2 \right) + \cdots + \alpha_q N \left( \phi_q s_{t-q}, \sigma_q^2 \right)
Mixture Transition Distributions (MTD) filter:

\[ \hat{s}_t = E(s_t \mid S^{t-1}) = - \sum_{i=1}^{p} (\alpha_0 g_i - \alpha_i \phi_i) s_{t-i} \]

\[ r_t = s_t - \hat{s}_t \]

Filter = \(1, \alpha_0 g_1 + \alpha_1 \phi_1, \ldots, \alpha_0 g_p + \alpha_p \phi_p\)

\[ \alpha_0 = 1 \Rightarrow WL \]
MTD Deconvolution

- $p =$ filter length
  
  \[ q + 1 = \# \text{ mixture components} \]

- $\phi$’s: $\rho_{k,j} =$ k-lag correlation from j-component
  
  $g_k =$ WL filter from 1st component

  \[ \Rightarrow \text{ filter } \approx (1, \alpha_0 g_1 - \alpha_1 \rho_{1,1}, \ldots, \alpha_0 g_p - \alpha_p \rho_{p,p}) \]

- EM algorithm
Approximation

• $g_k$ from WL

• $\rho_{k,j} \approx \rho_k$

• $q \ll p$

• WL plus smaller optimization
Plots

• Dynamic range reduction of $g_t \ast w_t$

• Reduction in phase rms error $(g_t \text{ vs } w_t^{-1})$
Dynamic range reduction (%) vs. s/n ratio.

Amplitude

s/n

Dynamic range reduction (%)
II. Cosmology

- Cosmological models $\rightarrow$ random fields

- Random fields: homogeneous (or isotropic)
  
  \[ E R(r) = \mu, \quad E R(r_1) R(r_2) = \gamma(\cos \theta_{12}) \]

- Not always Gaussian
Questions:

Generate random field(s):

1. Homogeneous
   Non-Gaussian
   Prescribed marginal
   Prescribed correlation function
2. Homogeneous

Non-Gaussian

Prescribed correlation function

Prescribed three-point correlation function
3. Multidimensional homogeneous
Non-Gaussian

Prescribed marginals

Prescribed cross-correlation functions

[with R. Vio, W. Wamsteker (ESA)
& P. Andreani (MP Garching)]
Multidimensional homogeneous fields

- $\mathbf{R}(\mathbf{r}) = ( R_1(\mathbf{r}), ..., R_n(\mathbf{r}) )$

- $R_i = \text{different frequency backgrounds}$

- $R_1 = \text{radiation background, } R_2 = \text{source field}$
Homogeneity

- Mean vector \( \mu_i = \mathbb{E} R_i(\mathbf{r}) \rightarrow \mu_R \)

- Cross-correlation matrix
  \[
  \rho_{ij}(\cos \theta) = \text{Cov} \left( R_i(\mathbf{r}_1), R_j(\mathbf{r}_2) \right) \rightarrow \rho_R(\theta)
  \]

- Spectral representation: \( A_{\ell,m} \) uncorrelated
  \[
  R(\mathbf{r}) = \sum_{\ell, m} A_{\ell,m} Y_{\ell,m}(\mathbf{r}),
  \]
  \[
  \mathbb{E} \left[ A_{\ell,m} A_{\ell,m}^* \right] = S_\ell
  \]
Defining Gaussian $G(r)$

- Just define

  Mean vector $\mu_G$

  Cross-correlation matrix $\rho_G(\theta)$

- Use spectral representation
Defining non-Gaussian $\mathbf{R}(\mathbf{r})$

- Much harder

  Marginal distributions $F_i$

  Mean vector $\mathbf{\mu}_\mathbf{R}$

  Cross-correlation matrix $\mathbf{\rho}_\mathbf{R}(\theta)$

- Start with Gaussian
Transforming Gaussian fields

- $\mathbf{G}(\mathbf{r})$ homogeneous Gaussian
  
  \[ \mathbf{R}(\mathbf{r}) = T(\mathbf{G}(\mathbf{r})) \text{ homogeneous} \]

- Marginals define $T$: 
  
  \[ T_i = F_i^{-1} \circ \Phi \]

- Find $\boldsymbol{\rho}_\mathbf{G}$ that maps to $\boldsymbol{\rho}_\mathbf{R}$
  
  \[ \rho_{R_i,R_j}(\theta) = H_{ij} \left( \rho_{G_i,G_j}(\theta) \right) \]
Properties of $H(\rho)$

- $H(0) = 0$

- Continuous and monotonically increasing on $[-1, 1]$

- Smooth on $(-1, 1)$

- Range $\subset [-1, 1]$
\[ \rho_{R_{ii}} \]
Recipe for Gaussian $G(r)$

- Invert for $\rho_G$

- Determine the cross-spectra $\Sigma_\ell$ from $\rho_G$.

- Generate Gaussian $A_\ell$ characterized by $\Sigma_\ell$

- Transform back
Example

• $T_1 = \text{Far-IR from ISO (Gaussian)}$

• $T_2 = \text{Source field (Gaussian mixture)}$

$$\rho(\tau) = \begin{pmatrix} \rho_0(\tau) & \alpha \rho_0(\tau) \\ \alpha \rho_0(\tau) & B(\tau) \end{pmatrix}$$

$\rho_0$ from cosmology, $B$ instrument’s smoothing
Summary

- Mixtures to adapt to nonstationarities

  ➤ Computational and physical issues

  ➤ Modeling uncertainties