Parameterizing Meshes with Applications

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Overview

• Motivation

• Parameterizing meshes
  – projection methods
  – linear energy methods
  – nonlinear methods

• Applications
  – texture mapping
  – quadrilateral remeshes and surface fitting
  – regular remeshes and hierarchies

• Summary

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Motivation (1)

- Analysis on surfaces is usually performed in Euclidean plane, using appropriate (local) coordinates.
  ⇒ one has to assign to every surface point a parameter value in the plane

- The result of the analysis often depends on the choice of the parameterization.

- To obtain good results use good parameterizations!

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Example: B–Spline interpolation

uniform  chord length  centripetal

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Q: What is a good parameterization?

A: One that preserve all the (basic) geometry length, angles, area, ...

⇒ **isometric parameterization**

**but**: possible only for developable surfaces e.g., there will always be distortion!

Try to keep the distortion as small as possible (change of length, area, angles,...)
Motivation (3)

Applications

Many operations, manipulations on/with surfaces require a parameterization as a preliminary step.

e.g.:

- texture mapping
- surface fitting
- hierarchical representations
- mesh conversion
Motivation: Applications of parameterizations

Texture mapping

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Motivation: Applications of parameterizations

Surface fitting

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Motivation: Applications of parameterizations
Parameterization of 3D data points

**Problem:**

For a triangulated set of data points

\[ P_i \in \mathbb{R}^3, T_j = \Delta(P_{j_0}, P_{j_1}, P_{j_2}) \]

find a planar parametrization

\[ p_i \in \mathbb{R}^2, t_j = \Delta(p_{j_0}, p_{j_1}, p_{j_2}) \]

with minimal distortion.

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Parameterizations: Projection methods

- Project data points onto a suitable plane
  - e.g. least square fitting plane of the data points
- very simple, works well only for planar geometries
- Generalization (Ma, Kruth: CAD 1995)
  Project on other simple surfaces,
  e.g. Coons patches
Parameterizations: Linear energy methods

\[ E = \frac{1}{2} \sum_{\{i,j\} \in \text{Edges}} c_{ij} \|p_i - p_j\|^2 \]

linear spring–energy:

Hooke’s law

shape–preserving

Floater ’96

harmonic energy

Pinkall/Polthier ’93
Eck et.al. SIGGRAPH ’95

Note: boundary points have to be fixed!

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Parameterization: Linear methods

\[ E = \frac{1}{2} \sum_{\{i,j\} \in \text{Edges}} c_{ij} \| p_i - p_j \|^2 \]

- Spring constants for Hooke:
  - chord length: \[ c_{ij} = \| P_i - P_j \|^{-1} \]
  - centripetal: \[ c_{ij} = \| P_i - P_j \|^{-1/2} \]
  - uniform: \[ c_{ij} = 1 \]

- harmonic energy:
  \[ E_D(f) := \frac{1}{2} \int_M \| \nabla f \|^2 \, dA \]

  minimize Dirichlet energy
  \[ c_{ij} = \cot(\alpha) + \cot(\beta) \]
  -- negative weights ? --

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Parameterization: Linear methods

- **shape-preserving**
  solve the linear system with positive weights \( \lambda_{ij} \) that sum to 1.

- **modifications**: more realistic energy functional
  e.g. \( E = \sum_{[i,j]\in \text{edges}} c_{ij} (||p_i - p_j|| - \ell_{ij})^2 \)

  \( \Rightarrow \) better results, but no longer linear!
**FACT:**

Any parametrization deforms the shape of the triangles.  
except for developable surfaces (e.g., planes, cylinders, conics)

**QUESTION:**

How to measure this deformation?
**Systematic approach:** How to measure the distortion?

**IDEA:**

Find a functional that measures the distortion of the *atomic linear maps*

\[ f_j : T_j \rightarrow t_j \]

and minimize

\[ \sum_j E(f_j) \]

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Systematic approach: properties of functional

The deformation functional should be invariant to

1. translations,
2. orthogonal transformations,
3. scalings,

and should avoid degeneracies by

4. punishing collapsing triangles very badly.

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**Systematic approach:** derivation of functional

Consider the atomic linear map

\[ g : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad x \mapsto Ax + b \]

**Properties:**

1. translations
   - ignore \( b \)

2. orthogonal transf.
   - ignore \( U \) and \( V \)

3. scalings
4. collapsing triangles

**Singular value decomposition**

\[ U^TAV = \Sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \]

**2–norm condition**

\[ \kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_2} \]
Drawback of the 2−norm condition

**PROBLEM:**

2−norm condition not easy to compute

**SOLUTION:**

Use *Frobenius Norm* condition instead

\[
\kappa_F(A) = \|A\|_F \|A^{-1}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} \sqrt{1/\sigma_1^2 + 1/\sigma_2^2}
\]

\[
= \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \kappa_2(A) + \frac{1}{\kappa_2(A)}
\]

\[
= \frac{\text{trace}(A^t A)}{\det A}
\]

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Minimization of the deformation functional

Use a ”Gauss–Seidel–like” method:

pick successively one vertex and position it optimal.
Parameterization: References & further methods


Maillot, Yahia, Verroust: Interactive texture mapping, in: Proceedings Siggraph ’93, 27–34

Ma, Kruth: Parameterization of randomly measured points for least squares fitting of B–spline curves and surfaces, CAD, vol. 27, 1995, 663–675


Greiner, Hormann: Interpolating and approximating scattered 3D data with hierarchical tensor product B–splines, in: Surface Fitting and Multiresolution Methods, 1997, 163–172


Duchamp, Certain, DeRose and Stuetzle: Hierarchical computation of PL harmonic embeddings, preprint, 1997


Sheffer, de Sturler: Surface parameterization for meshing by triangulation flattening, in: Proceedings 9th International Meshing Roundtable, 2000, 161–172


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Comparison

- linear methods can be realized fast and easy
- harmonic energy can produce foldovers (neg. weights)
- nonlinear methods (realistic springs, MIPS) often (not always) produce better results
- most methods require to specify a parameterization of the boundary, except MIPS

- The optimal parameterization method has not been found yet, or it does not exist!

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For large data sets (> 20k points): hierarchical approach

**Parametrizations: Hierarchical methods**

- **build mesh hierarchy**
- **optimize each level**

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Applications: texture mapping

texture mapping of uniform rectangular grid

real spring  MIPS  harmonic

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Applications: texture mapping

- gray-coded deformation energy per triangle
- texture mapping of uniform rectangular grid

MIPS

harmonic

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Applications: Quadrilateral Remeshing

Parameterizing triangle meshes

Quadrilateral remeshing and surface reconstruction

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The Problem of Badly Shaped Iso-Curves

**Problem:** smooth surface $s$, but badly shaped iso-curves

**Task:** find $r$, such that the iso-curves of $s \circ r$ are smooth
Quadrilateral Remeshing of Smooth Surfaces

1) evaluate $s$ at the knots of a uniform 2D grid

2) use shape information of 3D grid to move knots of the 2D grid

3) let $r$ be the piecewise bilinear mapping between the 2D grids
Applying these Methods to Triangle Meshes

1) parameterize the triangle mesh

2) quadrilateral remeshing of the triangle mesh

3) interpolate the remesh with a TP–B–Spline
Quadrilateral Remeshing of Triangle Meshes (1)

1) find an initial quadrilateral
2) iteratively split each quadrilateral into four
3) use weighted barycentres for new vertex positions
Quadrilateral Remeshing of Triangle Meshes (2)

Weighted barycentres:

\[ p = \frac{w_N p_N + w_E p_E + w_S p_S + w_W p_W}{w_N + w_E + w_S + w_W} \]

How to compute the weights?

\[ w_E = \text{area} \triangle (p_E, p_{NE}, p_{SE}) + \]
\[ \text{area} \triangle (p_E, p_{EE}, p_{SE}) + \]
\[ \text{area} \triangle (p_E, p_{SE}, p) + \]
\[ \text{area} \triangle (p_E, p, p_{NE}) \]
Quadrilateral Remeshing of Triangle Meshes (3)

using uniform barycentres

using weighted barycentres

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Interpolating the Quadrilateral Remesh

Computing the interpolating TP–B–Spline \( t \)

Vertices of the quadrilateral remesh: \( P_{00},...,P_{mn} \)

Uniform knot vectors: \( u = \{0,0,0,0,1,2,...,m−1,m,m,m,m,m\} \)
\( v = \{0,0,0,0,1,2,...,n−1,n,n,n,n,n\} \)

Interpolation conditions: \( t(i,j) = P_{ij} \)

Natural end conditions: \( t'' = 0 \) along the boundary

Requires solving \( (m+3) + (n+3) \) linear, tridiagonal systems
Examples (1)
Examples (2)
Examples (3)

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Application: regular remeshes and hierarchies

irregular triangle mesh of arbitrary topology

Decimation → Parameterization → Subdivision → Remeshing

Wavelet Techniques

hierarchy of [regular] triangle meshes, approximating the input mesh, stored in a compact way as wavelet coefficients
Application: regular remeshes and hierarchies

- Fine input mesh → Parameterized fine mesh → Regular remesh
- Coarse base mesh → Refined base mesh → Wavelet reconstruction

Decimation → Parameterization → Remeshing → Wavelet Techniques

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Application: regular remeshes and hierarchies

– we know very well how to parameterize disk–like objects
– but how to parameterize objects with arbitrary topology?

| dissect the object into disk–like patches and parameterize each patch separately |

– for each edge in the coarse mesh, find a corresponding path in the fine mesh
– for each triangle in the coarse mesh, parameterize the corresponding region of the fine mesh
Application: regular remeshes and hierarchies

- refine the base mesh by successive 1-to-4-splits

- use the parameterization to lift the refined base mesh onto the surface of the input mesh

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Regular remeshes and hierarchies: Example

- Fine input mesh
- Parameterized fine mesh
- Regular remesh
- Coarse base mesh
- Refined base mesh
- Wavelet reconstruction

Decimation
Parameterization
Remeshing
Wavelet Techniques

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