How Intractable is the “Invisible Hand”: Polynomial Time Algorithms for Market Equilibria

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Market Equilibrium

- People want to maximize happiness
- Find prices s.t. market clears
Walras, 1874

- Pioneered mathematical theory of general economic equilibrium
Arrow-Debreu Theorem, 1954

- Celebrated theorem in Mathematical Economics
- Shows existence of equilibrium prices using Kakutani’s fixed point theorem
Arrow-Debreu Theorem is highly non-constructive

- How do markets find equilibria?
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  Deng, Papadimitriou & Safra, 2002: linear case in P?
Market Equilibrium

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History

- Irving Fisher  1891 (concave functions)
  - Hydraulic apparatus for calculating equilibrium
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- V. 2002: alg for generalization of linear case
Market Equilibrium

• $n$ buyers, with specified money,
• $m$ goods (unit amount)
• Linear utilities: $u_{ij}$ utility derived by $i$ on obtaining one unit of $j$

$$U_i = \sum_j u_{ij}x_{ij}$$
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- Find prices s.t. market clears
Bang per buck

$100

$60

$20

$140

utilities
Bang per buck

$100 10 20 $20 10/20

$60 20 $40 20/40

$20 4 $10 4/10

$140 2 $60 2/60
Bang per buck

Given prices $p_j$, each $i$ picks goods to maximize her bang per buck, i.e.,

$$\alpha_i = \max_j \left\{ \frac{u_{ij}}{p_j} \right\}$$
for all $i$: most desirable $j$'s
• Any goods sold in equality subgraph make agents happiest

• How do we maximize sales in equality subgraph?
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• How do we maximize sales in equality subgraph?

**Use max-flow!**
Max flow

infinite capacities
Max flow
Idea of Algorithm

Invariant: source edges form min-cut
(agents have surplus)

Want: prices s.t. sink edges also form min-cut

Gradually raise prices, decrease surplus, until 0
ensuring Invariant initially

- Set each price to $1/n$
- Assume buyers’ money integral
How to raise prices?

• Ensure equality edges retained

\[ \frac{u_{ij}}{p_j} = \frac{u_{il}}{p_l} \]
How to raise prices?

• Ensure equality edges retained

\[ \frac{u_{ij}}{p_j} = \frac{u_{il}}{p_l} \]

• Raise prices proportionately

\[ \frac{p_j}{p_l} = \frac{u_{ij}}{u_{il}} \]
initialize: $x = 1$
$x = 2$: another min-cut

$x > 2$: Invariant violated
reinitialize: $x = 1$
unfreeze
$x = 1, \quad x \uparrow$
buyers

\[ m \]

goods
buyers

\[ m \]

equality

subgraph

\[ p \]

goods

ensure

Invariant
\[ m \quad \quad \quad p_x \]

\[ x = 1, \quad x \uparrow \]
\[ \Gamma(S) \{ \} S \]

\[ x \cdot p(S) = m(\Gamma(S)) \]
$\Gamma(S)$  \{  \}  \{  \}  $S$  

$x \cdot p(S) = m(\Gamma(S)) \implies \text{freeze } S$
prices in $S$ are market clearing
\[ \Gamma(S) \quad \text{frozen} \]

\[ S \quad \text{active} \]

\[ x = 1, \quad x \uparrow \]
$\Gamma(S)$ \hspace{1cm} $S$ \hspace{1cm} \text{frozen}

\text{active}

$p_x$

$x = 1, \quad x \uparrow$
\( \Gamma(S) \) | \( S \) | frozen
---|--|--
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active

\[ px \]

\[ x = 1, \quad x \uparrow \]
new edge enters equality subgraph
unfreeze component

frozen

active
• All goods frozen => terminate

(market Clears)
• All goods frozen => terminate
  (market clears)

• When does a new set go tight?
\[ x^* := \min_{\emptyset \neq S \subseteq A} \left\{ \frac{m(\Gamma(S))}{p(S)} \right\} \]
Try $S = A$ (all goods)

Let $x = \frac{m(B)}{p(A)}$.

Clearly, $x \geq x^*$

If $s$ is min-cut, $x = x^*$ and $S^* = A$. 
Otherwise, $x > x^*$. 
Sufficient to recurse on smaller graph
Termination

• Prices in $S^*$ have denominators $\leq \Delta = nU^n$,
  
  \[ U = \max_{ij} \{u_{ij}\} \]

• Terminates in $Mn^2 \Delta^2$ max-flows.
Polynomial time

• Pre-emptively freeze sets that have small surplus (at most $\mathcal{E}$).
\[ \Gamma(S^*) \quad S^* \]

freeze
add $\varepsilon$ to prices and find new min-cut
• Next freezing: prices must increase $\geq \varepsilon$.

• Problem: at end, surplus $\neq 0$. 
• Next freezing: prices must increase $\geq \varepsilon$.

• **Problem:** at end, surplus $\neq 0$.

• But, surplus $\leq n\varepsilon$. 
initial surplus

$M$
\[ \varepsilon = \frac{M}{2n} \]
final surplus \leq n\varepsilon = \frac{M}{2}
Polynomial time

Theorem: $O(n^2 (n \log U + \log Mn^2))$

max-flow computations suffice.
Question

• Is main algorithm, i.e., without pre-emptive freezing, polynomial time?
Question

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• Strongly polynomial?
Post-mortem
Primal-Dual Schema

Highly successful algorithm design technique from exact and approximation algorithms
Central Idea Behind Primal-Dual Schema

Two processes making local improvements (relative to each other) and achieving global objective
Post-mortem

- “primal” variables: flow in equality subgraph
- “dual” variables: prices
- algorithm: primal & dual improvements
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- “primal” variables: flow in equality subgraph
- “dual” variables: prices
- algorithm: primal & dual improvements
- nonzero flow from $j$ to $i$ \(\Rightarrow p_j = \frac{u_{ij}}{\alpha_i}\)
Post-mortem

- “primal” variables: flow in equality subgraph
- “dual” variables: prices
- algorithm: primal & dual improvements
- nonzero flow from $j$ to $i$ $\Rightarrow p_j = u_{ij} / \alpha_i$

“complementary slackness condition”
Post-mortem

- “primal” variables: flow in equality subgraph
- “dual” variables: prices
- algorithm: primal & dual improvements
- algorithm inspired by Kuhn’s primal-dual algorithm for bipartite matching
“Primal-Dual-Type” Algorithms

• Formal mathematical setting?

• Analogous setting for primal-dual algorithms: LP-Duality Theory
Concave utilities

- Buyers get satiated by goods
- Fix prices => each buyer has unique optimal bundle
Concave utilities

• Buyers get satiated by goods

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Economy of communication!

Distributed market clearing
Concave utility function

utility

amount of $j$
Piece-wise linear, concave

utility

amount of $j$
PTAS for concave fn.
Piece-wise linear, concave

utility

amount of $j$
Differentiate

\[ \text{rate} = \frac{\text{utility}}{\text{amount of } j} \]
$f_{ij}$ for buyer $i$, good $j$

rate

rate = utility/unit amount of $j$
Fix price of $j$, then utility/$\$ \text{ given by}$

\[
\frac{f_{ij}}{p_j}
\]

utility derived, \quad u(x) = \int_{0}^{x} \frac{f_{ij}(y)}{p_j} \, dy
fix price of $j$

utility

piecewise-linear, concave
V. 2002:
Rate at which \( i \) derives happiness depends on fraction of budget spent on \( j \).
Theorem: Equilibrium price is unique, and can be computed in polynomial time
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(Not unique in traditional model, even for piece-wise linear case)
Can generalize notion of “market clearing” -- assume that buyers have utility for money.
Does the spending constraint model measure up to traditional theory?
$f_{ij}$ for buyer $i$, good $j$

rate

$\$100$

continuous, decreasing rate function
utility derived, \[ u(x) = \int_0^x \frac{f_{ij}(y)}{p_j} dy \]

Strictly concave function.

Each buyer has unique optimal bundle.
Devanur & V., 2003

Equilibrium exists for cts., decreasing rate fns. (Proof uses Brauwer’s fixed point theorem),
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Devanur & V., 2003

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PTAS for computing equilibrium.
Devanur & V., 2003

Extend model to Arrow-Debreu setting.
Devanur & V., 2003

Extend model to Arrow-Debreu setting.

Equilibrium exists.

(Proof uses Kakutani’s fixed point theorem.)
Algorithmic Game Theory

• Mechanism design: find equilibria that ensure truthful, fair functioning of agents, and are efficiently computable.

• Approximations: deal with NP-hardness, stringent game-theoretic notions
Q: Distributed algorithm for equilibria?

• Appropriate model?

• Primal-dual schema operates via local improvements
• Global optimality via local improvement

• Exploit in distributed setting

Kelly, Low, Lapsley, Doyle, Paganini ...
TCP congestion control

primal process: packet rates at sources

dual process: packet drop at links

AIMD + RED solves utility maximization problem in limit
Kelly, ’97: charging, rate control and routing for elastic traffic

Kelly & V. 2002:
It is essentially a market equilibrium question, and generalizes Fisher’s problem!
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Q: Polynomial time alg?
Develop an algorithmic theory of market equilibria, via polynomial time exact and approximation algorithms
• w.r.t. prices $p$, $i$ sorts segments according to utility/\$, and partitions into classes
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Assume $i$ has $100$
• w.r.t. prices $p$, $i$ sorts segments according to utility/$$, and partitions into classes

$\text{forced} \quad \uparrow \quad \text{flexible} \quad \text{undesirable}$

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• **Invariant 1:** Approach eq. from below

• **Invariant 2:** Forced allocations follow sorted order
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  – simultaneously, for all $i$
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  – *simultaneously*, for all $i$

  – as prices change, allocs may become undesirable
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  Deallocate
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Deallocate - **exponential time??**
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  Deallocate - **exponential time**??

  Reduce prices
• **Invariant 1:** Approach eq. from below

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Deallocate - exponential time??

Reduce prices - measure of progress??
Algorithm

Maintains both Invariants

- deallocations

- monotonicity of prices
forced  flexible  undesirable
Can ensure prices