The Mathieu equation with complex parameters

Consider the Mathieu equation

$$w'' + (\lambda - 2h^2 \cos(2z)) w = 0.$$ 

For $h^2 \in \mathbb{R}$, let

$$\lambda_1(h^2) < \lambda_2(h^2) < \lambda_3(h^2) < \ldots$$

denote the eigenvalues of $\lambda$ such that the Mathieu equation admits nontrivial odd solutions with period $\pi$. The functions $\lambda_m$ are real-analytic. What happens if we continue these functions analytically into the complex $h^2$-plane?

The eigenvalue functions can be continued analytically throughout the complex $h^2$-plane when we avoid a certain countable set that has no finite point of accumulation ($h^2$-coordinates of branch points.) The eigenvalue functions approach a finite value at these branch points.
Since the eigenvalue function $\lambda_n(h^2)$ is analytic, it can be expanded in a power series at $h^2 = 0$. Let $\rho_n$ be the radius of convergence of this expansion. What is the growth order of $\rho_n$ as $n \to \infty$? It is easy to show that $\rho_n$ grows at least linearly with $n$. F. W. Schäfke conjectured that the $\rho_n$ grow like $n^2$ as $n \to \infty$. I proved that

$$\liminf \frac{\rho_n}{n^2} \geq kk'K^2 = 2.0418\ldots,$$

where the modulus $k$ of the complete elliptic integrals is determined by $2E = K$. 
Call two positive integers \( m, n \) equivalent, if there exists a curve starting at \( h^2 = 0 \) and ending at \( h^2 = 0 \) such that \( \lambda_m(h^2) \) when continued analytically along the curve becomes \( \lambda_n(h^2) \). F. W. Schäfke proved that there is only one equivalence class for the Mathieu equation. Let us look at the proof.

For \( k \in \mathbb{N} \), define

\[ \tau_k = \min_{n > k} \rho_n. \]

**Lemma 1** Let \( M \) be an equivalence class and \( k \in \mathbb{N} \). Then the function

\[ \sigma_k(h^2) = \sum_{n \in M, n \leq k} \lambda_n(h^2) \]

which is analytic in a neighborhood of \( \mu = 0 \) can be extended analytically onto the disk \( \{ h^2 : |h^2| < \tau_k \} \).

**Proof:** Let \( \phi(t) \) be a curve starting at \( h^2 = 0 \) and ending at \( h^2 = 0 \) with \( |\phi(t)| < \tau_k \) and which does not pass through branch points. Then analytic extension
along $\phi$ defines a bijection $\pi : M \to M$. We have $\pi(m) = m$ for $m > k$. This shows that $\sigma_k$ is one-valued in the disk apart from the finitely many branch points. Since these points are removable singularities for $\sigma_k$, the lemma is proved.

**Lemma 2** Let $M, k, \sigma_k$ be as in Lemma 1. Then

$$|\sigma_k''(0)| \leq 8k \frac{1}{\tau_k}$$

**Proof** Note that eigenpairs $(\lambda, \mu)$ satisfy

$$|\text{Im } \lambda| \leq 2|h^2|.$$

Therefore

$$|\text{Im } \sigma_k(h^2)| \leq 2k \tau_k$$

for $|h^2| < \tau_k$. By Lemma 1 and a modified form of Cauchy’s estimate, we obtain the statement of the lemma.

From Lemma 2, we obtain
Theorem 3 Let $M$ be an equivalence class. Then

$$\sum_{n \in M} \lambda''(0) = 0.$$ 

Now $\lambda''(0) < 0$ and $\lambda''(0) > 0$ for all $n \geq 2$. Therefore $N$ is the only equivalence class.