Integral relations for solutions of Hill’s equation

Consider the Hill equation
\[ u'' - g(x)u = 0, \]
where \( g : \mathbb{R} \rightarrow \mathbb{C} \) is even, continuous and of period \( \omega \). We assume that there is a nontrivial solution \( u_1 \) of period \( 2\omega \). Let \( u_2 \) be a linearly independent solution. The product \( v(x, y) = u_1(x)u_2(y) \) satisfies the partial differential equation
\[ (\partial_x^2 - \partial_y^2)v - (g(x) - g(y))v = 0. \]

Let \( A : \mathbb{R}^4 \rightarrow \mathbb{C} \) be the Riemann function of this equation, that is, for \((x_0, y_0) \in \mathbb{R}^2\) let \( A(\cdot, \cdot, x_0, y_0) \) be the unique solution with
\[ A(x, y, x_0, y_0) = 1 \quad \text{if} \quad y - y_0 = \pm(x - x_0). \]

Then
\[ \int_K (A\partial_y v - v\partial_y A) \, dx + (A\partial_x v - v\partial_x A) \, dy = 0 \]
where \( K \) is any closed rectifiable curve in \( \mathbb{R}^2 \).
We choose for $K$ the following pentagon
Theorem. There is a constant $\mu$ such that, for all $x_1, x_2, x_3 \in \mathbb{R}$,

$$\mu u_1(x_1)u_1(x_2)u_1(x_3) = \int_{-\omega}^{\omega} A(x, x_1, x_2, x_3)u_1(x) \, dx.$$ 

The constant $\mu$ can be expressed by

$$\mu = \frac{2\sigma \tau}{W[u_1, u_2]}$$

where $\sigma, \tau$ are determined by

$$u_1(x+\omega) = \sigma u_1(x), \quad u_2(x+\omega) = \tau u_1(x) + \sigma u_2(x).$$
For the Mathieu equation
\[ u'' + (\lambda - 2h^2 \cos(2x))u = 0 \]
we have
\[ A(x, y, x_0, y_0) = J_0((4h^2(F_1^2 - F_2^2)^{1/2}), \]
\[ F_1 = \cos x \cos y - \cos x_0 \cos y_0, \]
\[ F_2 = \sin x \sin y - \sin x_0 \sin y_0. \]

For the Lamé equation
\[ u'' + (\lambda - \nu(\nu + 1)k^2 \text{sn}^2 x)u = 0 \]
we have
\[ A(x, y, x_0, y_0) = P_\nu(G_1 + G_2 + G_3), \]
\[ G_1 = k^2 \text{sn} x \text{sn} x_0 \text{sn} y \text{sn} y_0, \]
\[ G_2 = -\frac{k^2}{k'^2} \text{cn} x \text{cn} x_0 \text{cn} y \text{cn} y_0, \]
\[ G_3 = \frac{1}{k'^2} \text{dn} x \text{dn} x_0 \text{dn} y \text{dn} y_0. \]