Optimization models for the financial valuation of supply chain risks

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Supply Chain Risk

- Demand/supply is variable in time, quantity, and quality.
- Inventories accumulate — product, equipment, personnel, even financial obligations.
- Profits are variable, ie RISKY.
Supply Chain Risk

• Demand/supply is variable in time, quantity, and quality.
• Inventories accumulate — product, equipment, personnel, even financial obligations.
• Profits are variable, i.e. RISKY.

Semiconductor Fab:

• 0-3 month WIP forecast has std. dev. ~ 75% of mean, and order/qualify machine tools takes 1-2 years.
• How to measure and compensate Fab’s risk exposures?
Basic approach to financial valuation:

- Observe market prices in related contracts.
- Hedge contract risk by trading “equivalents”.
- Market price of risk (~ shareholders’ risk).
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- Market price of risk (≈ shareholders’ risk).

Complications for Supply Chain risk valuation:

- Role of “non-market” information.
- Non-financial incentives and risks (market share, service-levels,...)
- Negotiation.
- Meta-effects: state-transitions due to scale, chunking and complexity, stochastics, industrial organization, etc.
OBJECTIVE: Present *optimization approach* to financial valuation and explore its use as tool in valuation of supply chain risks.

- “Supply chain” view of options pricing.
- Application to pricing risk in quantity-flexible contracting.
- Calibration to market.
- Final words
Options are used to manage financial risks... can one apply similar techniques to supply chain risk?
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Supply chain view of options pricing theory

- Option “payout” is function of “underlying”
- Trading in underlying can “(super)replicate” option payouts
- Price of option equals “cost of manufacturing” option payouts through trading.
Simple Options Pricing Example

- IBM stock price takes values \( \{Z_n^{IBM}\} \)
- IBM call option w/ strike $120 pays \( F_n = [Z_n^{IBM} - 120]^+ \)

\[
\begin{align*}
Z_1^{IBM} &= 200 \\
p_1 &= .99 \\
F_1 &= 80 \\
Z_0^{IBM} &= 100
\end{align*}
\]

option value = ??

\[
\begin{align*}
Z_2^{IBM} &= 50 \\
p_2 &= .01 \\
F_2 &= 0
\end{align*}
\]
Option price equals cost of trading strategy that “replicates” payoffs \( F_n = [Z_n^{IBM} - 120]^+ \)

Formulation as LP:

<table>
<thead>
<tr>
<th>asset</th>
<th>price</th>
<th>position</th>
</tr>
</thead>
<tbody>
<tr>
<td>stock:</td>
<td>( Z_n^{IBM} )</td>
<td>( \theta_n^{IBM} )</td>
</tr>
<tr>
<td>bond:</td>
<td>( Z_n^B = 1 )</td>
<td>( \theta_n^B )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\min_{(\theta)} \quad & F_0 \\
\text{st:} \quad & Z_0 \cdot \theta_0 = F_0 \\
& Z_n \cdot \theta_0 \geq F_n \quad (n = 1, 2)
\end{align*}
\]
\[
\begin{align*}
\max_{(q)} & \quad \sum_{n=1}^{2} q_n F_n \\
\text{st:} & \quad q_0 = 1 \\
& \quad q_n \geq 0 \quad (n = 1, 2) \\
& \quad q_0 = \sum_{n=1}^{2} q_n \quad \text{(bond)} \\
& \quad q_0 Z_0^{\text{IBM}} = \sum_{n=1}^{2} q_n Z_n^{\text{IBM}} \quad \text{(stock)}
\end{align*}
\]
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- $q$ are the weights of a probability measure $Q$
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\]

- \(q\) are the weights of a probability measure \(Q\)
- \(Q\) makes stock price \(\{Z_n^{\text{IBM}}\}\) into “martingale”
Dual of LP (Ross, 1996)

\[
\begin{align*}
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& \quad q_n \geq 0 \quad (n = 1, 2) \\
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& \quad q_0 Z_0^{\text{IBM}} = \sum_{n=1}^{2} q_n Z_n^{\text{IBM}} \quad (\text{stock})
\end{align*}
\]

- \(q\) are the weights of a probability measure \(Q\)
- \(Q\) makes stock price \(\{Z_n^{\text{IBM}}\}\) into “martingale”
- Option value is \(\max_{Q} \mathbb{E}_{Q} [ F_T ]\)
Martingale for Simple Example

There is only one possible solution to Dual.

\[ Z_0^{\text{IBM}} = 100 \]

\[ \text{option value} = \frac{1}{3}80 + \frac{2}{3}0 \]

\[ p_1 = 0.99 \]
\[ q_1 = \frac{1}{3} \]
\[ Z_1^{\text{IBM}} = 200 \]
\[ F_1 = 80 \]

\[ p_2 = 0.01 \]
\[ q_2 = \frac{2}{3} \]
\[ Z_2^{\text{IBM}} = 50 \]
\[ F_2 = 0 \]
Replicating Portfolio for Simple Example

Determine replicating portfolio from knowledge of dual optimal value

\[
\begin{align*}
\theta^B_0 + 100 \theta^{IBM}_0 &= 80/3 \\
\theta^B_0 + 200 \theta^{IBM}_0 &\geq 80 \quad (n = 1) \\
\theta^B_0 + 50 \theta^{IBM}_0 &\geq 0 \quad (n = 2)
\end{align*}
\]

Solution:

<table>
<thead>
<tr>
<th>asset</th>
<th>position</th>
</tr>
</thead>
<tbody>
<tr>
<td>stock:</td>
<td>( \theta^{IBM}_0 = 160/300 ) shares</td>
</tr>
<tr>
<td>bond:</td>
<td>( \theta^B_0 = -80/3 ) loan</td>
</tr>
</tbody>
</table>
Replicating Portfolio for Simple Example

\[ p_1 = 0.99 \]
\[ q_1 = \frac{1}{3} \]
\[ \theta_0^{\text{IBM}} = \frac{160}{300} \]
\[ \theta_0^B = -\frac{80}{3} \]
\[ \text{option value} = \frac{80}{3} \]

\[ p_2 = 0.01 \]
\[ q_2 = \frac{2}{3} \]

\[ Z_1^{\text{IBM}} = 200 \]
\[ F_1 = 80 \]
\[ 200 \times \frac{160}{300} - \frac{80}{3} = 80 \]

\[ Z_2^{\text{IBM}} = 50 \]
\[ F_2 = 0 \]
\[ 50 \times \frac{160}{300} - \frac{80}{3} = 0 \]
General setup for discrete probability space

- Vector stochastic price process $\{Z_t\}_{t=0}^{T}$, with $Z^0_t := 1$ as the “bond”.
- Distinct paths at time $t$ correspond to nodes $n \in \mathcal{N}_t$, with parent node $a(n) \in \mathcal{N}_{t-1}$ and child nodes $c(n) \subset \mathcal{N}_{t+1}$.
- Expectation: $E^Q[ Z_t ] := \sum_{n \in \mathcal{N}_t} q_n Z_n$
Using Stochastic Programming to (Super-)Replicate Claims

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Optimization model (super-)replicates cash flows \( F_n \) at minimum price

- portfolio holdings \( \theta_n \)
- trades \( \Delta \theta_n := \theta_n - \theta_{a(n)} \).

\[
\begin{align*}
\min_{\theta} & \quad F_0 \\
\text{st:} & \quad Z_0 \cdot \theta_0 = F_0 \\
& \quad Z_n \cdot \Delta \theta_n = -F_n \quad (n \in \mathcal{N}_t, t \geq 1) \\
& \quad Z_n \cdot \theta_n \geq 0 \quad (n \in \mathcal{N}_T)
\end{align*}
\]

This is called a "stochastic (linear) program".
**Fundamental Pricing Theorem**

**Theorem 1**  There is a trading strategy that super-replicates $F_n$ iff there exists a martingale probability measure $Q$, in which case the option price is

$$F_0^* = \max_{Q \in \text{MPM}} E^Q \left[ \sum_{t=1}^{T} F_t \right]$$  (2)

Proof is by strong LP duality. Here is the dual:

$$\max(q) \quad \sum_{n=1}^{N} q_n F_n$$

$$\text{st:} \quad q_0 = 1$$

$$q_n \geq 0 \quad (n \in \mathcal{N}_T)$$

$$q_n Z_n = \sum_{m \in \mathcal{C}(n)} q_m Z_m \quad (n \in \mathcal{N}_t, t = 0, \ldots, T - 1)$$

Details in (King, 1998). Continuous state extensions in (King-Korf, 2002).
Options Pricing is a Dual Method

The basic method as outlined in Harrison-Pliska (1981):

1. Describe stochastic price process \( \{Z_t\} \), with prices normalized so that the bond's value is 1 in all states.

2. Find a dual solution \( Q \), a probability measure satisfying martingale equalities

\[ Z_t = E^Q \left[ Z_{t+1} | Z_t \right] \]

3. Calculate payouts \( \{F_t\} \)

— may need to know \( Q \), eg, when \( F \) is an American-style option.

4. Option price is

\[ F_0 = E^Q \left[ \sum_t F_t \right] \]

5. (Super-)Replicating portfolio is

\[ \theta_0 = \partial Z_0 E^Q \left[ \sum_t F_t \right] \]
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5. (Super-)Replicating portfolio is \( \theta_0 = \partial Z_0 E^Q [ \sum_t F_t ] \)

This describes the seller, who receives \( F_0 \) in order to generate \( F_n \).

... but what about the buyer?
The Arbitrage Interval

Theorem 2  *The maximum price that the buyer will pay for stochastic cashflows* $F_n$ *is*

$$F_0^* = \min_{Q \in \text{MPM}} E^Q \left[ \sum_{t=1}^{T} F_t \right]$$  \hspace{1cm} (3)

Proof is by reversing signs in (2).
The Arbitrage Interval

**Theorem 3**  The maximum price that the buyer will pay for stochastic cashflows $F_n$ is

$$F_0^* = \min_{Q \in \text{MPM}} E^Q \left[ \sum_{t=1}^{T} F_t \right]$$

(4)

Proof is by reversing signs in (2).

Buyers and sellers have different prices:

- Seller’s minimum offering price is $F_0^w := \max_{Q \in \text{MPM}} E^Q \left[ \sum_{t=1}^{T} F_t \right]$
- Buyer’s maximum acceptable price is $F_0^b := \min_{Q \in \text{MPM}} E^Q \left[ \sum_{t=1}^{T} F_t \right]$
- One has $F_0^b < F_0^w$, unless $Q$ is unique.

**Arbitrage interval:** $[F_0^b, F_0^w]$. 
The Arbitrage Interval

**Theorem 4**  The maximum price that the buyer will pay for stochastic cashflows $F_n$ is

$$F_0^* = \min_{Q \in 	ext{MPM}} E^Q \left[ \sum_{t=1}^{T} F_t \right]$$  \hfill (5)

Proof is by reversing signs in (2).

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- One has $F_0^b < F_0^w$, unless $Q$ is unique.

“Arbitrage interval”: $[F_0^b, F_0^w]$.

... we conclude that neither seller nor buyer want to trade options!
Let $F_0$ be the market price of the claim.

- Investor *buys* if $F_0 < E^Q [ \sum_{t=1}^{T} F_t ]$
- Investor *sells* if $F_0 > E^Q [ \sum_{t=1}^{T} F_t ]$

Buyers and Sellers have *different* martingale measures.
Reasons to Trade Options

Let $F_0$ be the market price of the claim.

- Investor buys if $F_0 < E^Q \left[ \sum_{t=1}^{T} F_t \right]$
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Buyers and Sellers have different martingale measures.

Trading occurs because buyers and sellers differ:

1. Transactions costs and/or taxes.
2. Views of the risk of the option.
3. Endowments and liabilities.
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Buyers and Sellers have different martingale measures.

Trading occurs because buyers and sellers differ:

1. Transactions costs and/or taxes.
   
   . . . only makes the Arbitrage Interval wider.

2. Views of the risk of the option.

3. Endowments and liabilities.
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Buyers and Sellers have different martingale measures. Trading occurs because buyers and sellers differ:

1. Transactions costs and/or taxes.
2. Views of the risk of the option.
   
   ... no difference unless seller is willing to take a loss.
3. Endowments and liabilities.
Reasons to Trade Options

Let $F_0$ be the market price of the claim.

- Investor buys if $F_0 < E^Q [\sum_{t=1}^{T} F_t ]$
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Buyers and Sellers have different martingale measures. Trading occurs because buyers and sellers differ:

1. Transactions costs and/or taxes.
2. Views of the risk of the option.
3. Endowments and liabilities.

...the most important reasons to buy/sell options are

- differences in endowments: initial portfolios
- differences in liabilities: future cash flows correlated with the value of the underlying

Alan King - Math Sciences, IBM Research - IMA Sept 2002 – p.15/27
This optimization approach to options pricing appears in *(King, Math. Programming, 2002)*

General features are:

1. Options can be replicated iff the price process supports a Martingale Probability Measure.
2. Buyers and Sellers of options must have different MPM
3. Differences in MPM arise from differences in endowments and future liabilities.
Risk analysis of Quantity-Flexible Supply Contracts (Ahmed and King, 2002).

- QFS contract between single buyer, single seller.
- Fixed price per unit; quantity demanded is variable.
- Seller may pay penalties if out-of-stock.
- Examples: IBM Printer Division, Sun, HP.

Buyer pays “Franchise Fee” to compensate for seller’s risk and loss of pricing power.

...options pricing provides a guide to how to compute this fee.
Two parts to modeling QFS franchise fee:

1. Production to meet demand or pay unmet demand penalty
2. Trading in correlated securities to hedge risks

We will consider supplier’s point-of-view, but for simplicity we model only capacity expansion decisions.
Production model:

- Buyer's demand: \( \{d_t\}^T_{t=1} \)
- Supplier's marginal profit: \( s_t \)
- Shortage penalty: \( \gamma_t \)
- Capacity expansion charge: \( \alpha_t \)
- Capacity maintenance: \( \delta_t \)
- Capacity expansion decision: \( X_t \)
- Unmet demand variable: \( U_t \)

Market model: \( Z_t \) and \( \theta_t \), etc.
Supplier’s Stochastic Program

\[ \min_{\theta, x, u, v} \quad V \]
\[ \text{st:} \]
\[ Z_0 \theta_0 = V - \alpha_0 X_0 \]
\[ Z_n \theta_n = Z_n \theta_{a(n)} + s_n d_n - \alpha_n X_n \]
\[ -\gamma_n U_n - \delta_n \sum_{m \in A(n)} X_m \quad n \in \mathcal{N}_t, 1 \leq t \leq T - 1 \]
\[ Z_n \theta_n = Z_n \theta_{a(n)} + s_n d_n \]
\[ -\gamma_n U_n - \delta_n \sum_{m \in A(n)} X_m \quad n \in \mathcal{N}_T \]
\[ Z_n \theta_n \geq 0 \quad n \in \mathcal{N}_T \]
\[ \sum_{m \in A(n)} X_m + U_n \geq d_n \quad n \in \mathcal{N}_t, 1 \leq t \leq T \]
\[ X_n \geq 0 \quad n \in \mathcal{N}_t, 0 \leq t \leq T - 1 \]
\[ U_n \geq 0 \quad n \in \mathcal{N}_t, 1 \leq t \leq T \]
Martingale Assumption

Assume martingale measure $Q$ is known.

**Theorem 5**  The franchise fee is $F_0 = \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} d_n [v^*_n - q_n s_n]$ where $v^*$ is the optimal solution of the linear program

$$
\begin{align*}
\max & \quad \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} d_n v_n \\
\text{s.t.} & \quad \sum_{m\in\mathcal{D}(n)} v_n \leq C(n) \quad n \in \mathcal{N}_t, 0 \leq t \leq T - 1 \\
& \quad 0 \leq v_n \leq D(n) \quad n \in \mathcal{N}_t, 1 \leq t \leq T
\end{align*}
$$

(7)

with $C(n) := (q_n \alpha_n + \sum_{m\in\mathcal{D}(n)} q_m \delta_m)$, and $D(n) = q_n \gamma_n$. 

1. Initialize $v^*_n = 0$ for all $n$.
2. Sort $\{d_n\}$ such that $d_{n1} > d_{n2} > \cdots > d_{nK}$.
3. Repeat the following steps for $k = 1, \ldots, K$:
   - Let $\mathcal{N}_k = \{n|d_n = d_{nk}\}$,
   - Repeat the following steps until $\mathcal{N}_k = \emptyset$:
     - For each $n \in \mathcal{N}_k$, calculate
       \[
       v'_n = \min\{D(n), \min_{m \in \mathcal{Q}(n)} \{C(m) - \sum_{q \in \mathcal{D}(m)} v^*_q\}\}.
       \]
     - Let $m = \text{argmax}_{n \in \mathcal{N}_k} \{v'_n\}$, set $v^*_m = v'_m$ breaking ties arbitrarily, set $\mathcal{N}_k \leftarrow \mathcal{N}_k \setminus \{m\}$.
4. Return $v^*_n$, optimal solution.
Primal view of QFS

Take dual of in “market” variables, to isolate role of martingale measure

\[
\min_{X,U} \max_{q \in \mathcal{Q}} \beta_0 C_0 X_0 + \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n [C_n X_n + \gamma_n U_n - s_n d_n]
\]

s.t.

\[
\sum_{m \in A(n)} X_m + U_n \geq d_n \quad n \in \mathcal{N}_t, 1 \leq t \leq T
\]

\[
X_n \geq 0 \quad n \in \mathcal{N}_t, 0 \leq t \leq T - 1
\]

\[
U_n \geq 0 \quad n \in \mathcal{N}_t, 1 \leq t \leq T
\]

(8)

Shows role of \( Q \) in “risk-neutral discounting” of risky profits.

Note that fixing \( Q \) results in an underestimate of franchise fee.
Calibration of Martingale Measure

Where may we look for probability measure for this analysis?
... calibration (King and Penannen, 2002; see also Avellenadas and collaborators (1999-2002)) $F^i$, $i = 1, \ldots, k$ have bid/ask prices $F^i_b < F^i_a$, payoffs $F^i_n$.
Let $L_n$ be the “liability” cash flow in state $n$.

\[
\min_{v_+, v_-} \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n L_n + (v_+ + v_-)
\]
\[
\begin{align*}
q_0 & = 1, \\
q_n & \geq 0 \quad n \in \mathcal{N}_T, \\
\sum_{m \in \mathcal{C}(n)} y_m Z_m & = q_n Z_n \quad n \in \mathcal{N}_t, \ t = 1, \ldots, T - 1, \\
\sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n F_n & \leq F_a + v_+, \\
\sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n F_n & \geq F_b - v_-, \\
v_+, v_- & \geq 0
\end{align*}
\]
**Primal View of Calibration Model**

\[
\begin{align*}
\min_{\theta, \xi_+, \xi_-} & \quad V \\
\text{st:} & \quad Z_0 \cdot \theta_0 + (F_a \cdot \xi_+ - F_b \cdot \xi_-) = V \\
& \quad Z_n \cdot (\theta_n - \theta_{a(n)}) - F_n \cdot (\xi_+ - \xi_-) = L_n \quad n \in \mathcal{N}_t, \ t = 1, \ldots, T, \\
& \quad Z_n \cdot \theta_n \geq 0 \quad n \in \mathcal{N}_T, \\
& \quad \xi_+, \xi_- \in [0, 1].
\end{align*}
\]

1. The primal hedges the liability with smallest cash.
2. Market-traded options can be bought/sold, but positions are bounded.
3. The dual variable is a martingale probability measure.
4. The dual maximizes the integrated cash flow of the liabilities plus penalties for market-traded option prices being too far over the ask or too far under the bid.
Investing in market traded options automatically calibrates $Q$ — so long as these positions are bounded.

Surprise: there is no need to specify probability measure!

May not be able to correlate all QFS risks with market.

- $Q$ is martingale on market $\mathcal{M} \subset \mathcal{N}$
- could average production flows conditional on $\mathcal{M}$ and solve discounted calibration/QFS problem
- or could simply fix $q_n = q_m / |\mathcal{N}_t(m)| \forall n \in \mathcal{N}_t(m)$ for all “market” nodes $m \in \mathcal{M}$
Lots of people have helped us in this project

- Dave Jensen
- Samer Takriti
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- Olga Streltchenko
- Teemu Penannen
- R.T. Rockafellar
- Vasant Naik