Classification and Reformulation: a Direct Way to Tackle Multi-Item Lot-Sizing Problems by MIP

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Outline

- Decomposition of Multi-Item Lot-Sizing
- Four Instances
- Single Item Classification
- Single Item: Reformulation, Separation, Optimization
- Further Classification
- The Instances: Classification, Reformulation, Solution
- Further Results
- Open Questions
A Typical Multi-Item Lot-Sizing Problem

$ NI $ is number of items

$ NT $ is number of periods

$d^i_t$ is demand for $i$ in $t$

$p^i_t$ is unit production cost of $i$ in $t$

$h^i_t$ is unit storage cost per unit of $i$ in $t$

$f^i_t$ is fixed set-up cost if $i$ produced in $t$

$C^i_t$ is maximum production of $i$ in $t$

Single Machine - Capacity Constraints in Each Period,

Variables:

$x^i_t$ is production of $i$ in $t$

$s^i_t$ is stock of $i$ at end of $t$

$y^i_t = 1$ if set up to produce $i$ in $t$, $y^i_t = 0$ otherwise
Basic Formulation and Structure

\[
\begin{align*}
\min & \sum_{i,t} (p^i_t x_t^i + h^i_t s_t^i + f^i_t y_t^i) \\
& s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall i, t \\
& x_t^i \leq C_t^i y_t^i \quad \forall i, t \\
& \sum_i a^i x_t^i + \sum_i b^i y_t^i \leq L_t \quad \forall t \\
x, s \geq 0, y \in \{0, 1\}
\end{align*}
\]

What is the structure of the feasible region?

\[X = \left( \bigcap_{i=1}^{NI} Y^i \right) \cap \left( \bigcap_{t=1}^{NT} Z^t \right)\]

\(Y^i\) is the single item lot-sizing region

\[Y^i = \{(x, s, y) \in \mathbb{R}_+^{NT} \times \mathbb{R}_+^{NT} \times \{0, 1\}^{NT} : s_{t-1}^i + x_t^i = d_t^i + s_t^i, \ x_t^i \leq C_t^i y_t^i \ \forall t\}\]

\(Z^t\) is the single period resource constraint region

\[Z^t = \{(x_t, y_t) \in \mathbb{R}_+^{NI} \times \{0, 1\}^{NI} : \sum_i a^i x_t^i + \sum_i b^i y_t^i \leq L_t, x_t^i \leq C_t^i y_t^i \ \forall t\}\]

Strategy: Find or approximate \(\text{conv}(Y^i)\) and \(\text{conv}(Z^t)\).
Four Problems

Problem 1. Bottling lines

Four families of products. 30 day planning horizon. The line is available 16 hours per day, and only one family can be produced per day. There are storage, set-up and start-up costs.

\[
\begin{align*}
\min & \sum_{i,t} (p^i x^i_t + b^i y^i_t + f^i y^i_t + g^i z^i_t) \\
\text{s.t.} & \quad s^i_{t-1} + x^i_t = d^i_t + s^i_t \quad \forall i, t \\
& \quad x^i_t \leq C^i y^i_t \quad \forall i, t \\
& \quad \sum_i y^i_t \leq 1 \quad \forall t \\
& \quad z^i_t \geq y^i_t - y^i_{t-1} \quad \forall i, t \\
& \quad z^i_t \leq y^i_t \quad \forall i, t \\
& \quad x, s \geq 0, y, z \in \{0, 1\}
\end{align*}
\]

$z^i_t$ is a start-up variable. It takes value 1, if item $i$ is set up in $t$, but not in $t - 1$. 
Problem 2. Sequence-dependent Changeover Costs

Produce at full capacity, so demands can be normalized with $d^i \in \{0, 1\}$. 10 items and a 35 period planning horizon. There are storage and sequence-dependent changeover costs.

\[
\begin{align*}
\min & \sum_{i,t} h^i s^i_t + \sum_{i,j,t} q^{ij}_t x^{i,j}_t \\
& s^i_{t-1} + x^i_t = d^i_t + s^i_t \ \forall \ i, t \\
& x^i_t \leq y^i_t \ \forall \ i, t \\
& \sum_{i} y^i_t = 1 \ \forall \ t \\
& \chi^{ij}_t \geq y^i_{t-1} + y^j_t - 1 \ \forall \ i, j, t \\
& x, y \in \{0, 1\}, s, \chi \geq 0
\end{align*}
\]
**Problem 3. Pringles**

Production line - 30 products (6 flavours), 60 periods. Each item is produced at full capacity, and only one item is produced per period. Backlogging is allowed.

\[
\min \sum_{i,t} (b^i_t r^i_t + h^i_t s^i_t) \\
 s^i_{t-1} - r^i_{t-1} + C^i_t y^i_t = d^i_t + s^i_t - r^i_t \quad \forall i, t \\
\sum_i y^i_t \leq 1 \quad \forall t \\
s, r \geq 0, y \in \{0, 1\}
\]

+ constraints on the sequencing of flavours.
Problem 4: worms

\[
\begin{align*}
\min & \sum_{it} \alpha_t above_t^i + 10000 \sum_{it} late_t^i + \sum_{it} 0.25 below_t^i \\
& \sum_i b^i y_t^{ik} + \sum_i a^i x_t^{ik} \leq 2L_t/NK \\
(S_0^i) + inv_t^- + \sum_k x_t^{ik} = d_t^i + inv_t^i \\
inv_t^i = SS_t^i - below_t^i + above_t^i \\
x_t^{ik} \leq C_t^{ik} y_t^{ik} \\
x_t^{ik} \geq MINBATCH y_t^{ik} \\
below_t^i \leq SS_t^i \\
x, inv, below \geq 0, y \in \{0, 1\}
\end{align*}
\]
Classification: Single Item \((d_{tl} \equiv \sum_{j=t}^{l} d_j)\)

Lot-Sizing (LS)

\[
\begin{align*}
\min & \sum_t(k'_t + p'_t x_t + f_t y_t) \\
\text{s.t.} & \quad s_{t-1} + x_t = d_t + s_t \quad \forall t \\
& \quad x_t \leq C_t y_t \quad \forall t \\
& \quad x, s \geq 0, y \in \{0, 1\}
\end{align*}
\]

Wagner-Whitin (WW) \(p'_{t-1} + h'_{t-1} \geq h't \forall t\).

Produce as late as possible.

\[
\begin{align*}
\min & \sum_t(h_t s_t + f_t y_t) \\
\text{s.t.} & \quad s_{t-1} + \sum_{j=t}^{l} C_j y_j \geq d_{tl} \quad \forall t, l, t \leq l \\
& \quad s \geq 0, y \in \{0, 1\}
\end{align*}
\]

Discrete Lot-Sizing with Initial Stock (DLSI)

\(x_t = C_t y_t\) for \(t = 1, \ldots, n\).

\[
\begin{align*}
\min & \quad h_0 s_0 + \sum_t f_t y_t \\
\text{s.t.} & \quad s_0 + \sum_{j=1}^{l} C_j y_j \geq d_{1l} \quad \forall l \\
& \quad s_0 \geq 0, y \in \{0, 1\}
\end{align*}
\]
Discrete Lot-Sizing (DLS) \[ s_0 = 0 \]

\[ \sum_t f_t y_t \]

\[ \sum_{j=1}^l C_j y_j \geq d_{1l} \quad \forall l \]

\[ y \in \{0, 1\} \]

Production Classification: LS, or WW, or DLSI, or DLS

Capacity Classification

Varying Capacity (C): \( C_t \) varying

Constant Capacity (CC): \( C_t = C \) for all \( t \)

Uncapacitated (U): \( C_t \geq d_{tn} \) for all \( t \)

Complexity

DLS-C is NP-hard, so all four varying capacity problems are hard.

LS-CC is easy, so all eight CC and U problems are easy.
Further Single Item Variants

The third field $VAR$ concerns extensions/variants to one of the twelve problems $\{PROB\} - \{CAP\}$ considered so far.

- Backlogging (B).
- Start-Up Costs (SC).
- Start-Up Times (ST).
- Minimum Production Levels (LB).
- Sales (SL). $d_t$ is now an upper bound on the sales in period $t$.
- Safety Stocks (SS). There is a lower bound $s_t \geq S_t$.

Three fields for a single item lot-sizing problem

$\{LS, WW, DLSI, DLS\} - \{C, CC, U\} - \{B, SC, ST(C), SL, LB(C), SS\}^*$
Multi-Item Classification: Machines

Machines

\( NK \) is the number of machines

\( MI \) indicates that machines are identical, having the same production rate and capacity

\( C_{t}^{ik} = C_{t}^{i} \) for all \( k \), but possibly differing costs

\( MD \) indicates that the machines are different.

\( LT \) indicates that there are lead times.
Multi-Item Classification: Resource Constraints $Z^t$

Small time buckets (Short time periods) $SB[1, 2]$ small bucket model with at most one or at most two set-ups per period. $SB[1]$ is often referred to as a model with mode constraints.

Big time buckets (Longer time periods)
$BB[.]$ denotes a big bucket model with at least one joint capacity constraint imposing a limit $L_t^k$
$SET$ indicates that there are also set-up times.
$ST$ indicates that there are start-up times.
$SQT$ indicates that there are sequence dependent changeover times $q_t^{ijk}$.
$SQC$ indicates that there are sequence dependent changeover costs $c_{t}^{ijk}$.

Thus the machine classification looks like

$$\{NK = [.], [IM, VM, LT]: [SB(1, 2), BB(SET, ST, SQT), SQC]\}.$$
Resource Constraints $Z^t$: Modelling

Single Set-up $SB1$: 

$$Z^t = \{ y \in \{0, 1\}^N : \sum_i y_t^i \leq 1 \}$$

Two Set-ups $SB2$: Typical Sequence (2, 3)(3, 4)(4, 4)(4, 1)

Last item in period $t$ is first item in period $t + 1$

$BB - SET$

$$Z^t = \{(x, y) \in R^N_+ \times \{0, 1\}^N : \sum_i (a^i x^i + b^i y_t^i) \leq B_t, x \leq C_t y_t \ \forall t\}$$
\[ B B - S T \]

\[ Z^t = \{(x, y) \in R_+^{N_1} \times \{0, 1\}^{N_1} \times \{0, 1\}^{N_1} : \sum_i (a^i x^i_t + b^i y^i_t) \leq B_t, \]
\[ x^i_t \leq C^i_t y^i_t, z^i_t \leq y^i_t, z^i_t \geq y^i_t - y^i_{t-1} \text{ etc. } \forall i \} \]

\[ B B - S Q T \]

\[ Z^t = \{(x, y) \in R_+^{N_1} \times \{0, 1\}^{N_1} \times \{0, 1\}^{N_1} : \sum_i (a^i x^i_t + \sum_j \tau^{ij} \chi^i_{t}) \leq B_t, \]
\[ x^i_t \leq C^i_t y^i_t, \sum_i \chi^i_{t} = y^i_t, \text{ etc. } \forall i, j \} \]
Multi-Item Classification: Multi-Level Production

$NL$ denotes the number of levels, with $\rho_{t}^{ijk}$ the number of units of item $i$ needed to produce one item of $j$ on machine $k$ in period $t$ for each item $j \in S(i)$, the set of successors of $i$.

$G$ denotes a general product structure

$A$ denotes assembly structure

$S$ denotes in series assembly structure, i.e. linear.

The production structure classification is simple

\[ \{ NL = [.], [G, A, S] \}. \]
Reformulations $NI = 1, NT = n$

<table>
<thead>
<tr>
<th>FORMULATION</th>
<th>$LS$</th>
<th>$WW$</th>
<th>$DLSI$</th>
<th>$DLS$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>$SP ; O(n) \times O(n^2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$FL ; O(n^2) \times O(n^2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$WW ; O(n^2) \times O(n)$</td>
<td>$[19, 13]$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$CC$</td>
<td>$O(n^3) \times O(n^3)$</td>
<td>$[44]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$O(n^2) \times O(n^2)$</td>
<td>$[33]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$O(n) \times O(n)$</td>
<td>$[26, 33]$</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>$O(n) \times O(n)$</td>
<td>Folklore</td>
<td></td>
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</tr>
<tr>
<td>$U$</td>
<td>$(l, S)$</td>
<td>$WW$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$O(n \log n)$</td>
<td>$[4]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$CC$</td>
<td>$klSI^*$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$HEUR$</td>
<td>$[32]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$klSI(WW)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$O(n^2 \log n)$</td>
<td>$[33]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Mixing$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$O(n \log n)$</td>
<td>$[17, 26, 33]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Above$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U$</td>
<td>$O(n \log n)$</td>
<td>$[1, 14, 46]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$CC$</td>
<td>$O(n)$</td>
<td>$[1, 14, 46]$</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>$O(n \log n)$</td>
<td>$[44]$</td>
<td></td>
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<tr>
<td></td>
<td>$O(n)$</td>
<td></td>
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<tr>
<td></td>
<td>$O(n)$</td>
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<td></td>
</tr>
</tbody>
</table>

Table 1: Model $PROB \leftarrow \{CC, U\}$
## With Backlogging

<table>
<thead>
<tr>
<th>FORMULATION</th>
<th>LS</th>
<th>WW</th>
<th>DLSI</th>
<th>DLS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>$SP(B) O(n) \times O(n^2)$</td>
<td>$O(n^2) \times O(n)$</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td></td>
<td>$FL(B) O(n^2) \times O(n^2)$</td>
<td>[33]</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td><strong>CC</strong></td>
<td>$O(n^3) \times O(n^3)$</td>
<td>$O(n^2) \times O(n^2)$</td>
<td>$O(n^2) \times O(n^2)^*$</td>
<td>$O(n) \times O(n)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SEPARATION</th>
<th>$Ext(l, S)^*$</th>
<th>$Cycles$</th>
<th>$MCF, HSep(VV)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[31]</td>
<td>[33]</td>
<td></td>
</tr>
<tr>
<td><strong>CC</strong></td>
<td>$HEUR(Con)$</td>
<td>$FC, RC, GMix$</td>
<td>$GMix$</td>
</tr>
<tr>
<td></td>
<td>[28, 21, 26]</td>
<td>[26]</td>
<td>[26]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>OPTIMIZATION</th>
<th>$O(n \log n)$</th>
<th>$O(n)$</th>
<th>−</th>
<th>−</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[1, 14, 46]</td>
<td>[1, 14, 46]</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td><strong>CC</strong></td>
<td>$O(n^3)$</td>
<td>−</td>
<td>$O(n \log n)$</td>
<td>$O(n \log n)$</td>
</tr>
</tbody>
</table>

Table 2: Model $PROB - \{CC, U\} - B$ with Backlogging
### With Start-Ups

<table>
<thead>
<tr>
<th>FORMULATION</th>
<th>LS</th>
<th>WW</th>
<th>DLS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>$SP(ST) O(n^2) \times O(n^2)$</td>
<td>$O(n^2) \times O(n)$</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$FL(ST) O(n^3) \times O(n^2)$</td>
<td>[48], [43]</td>
<td>–</td>
</tr>
<tr>
<td><strong>CC</strong></td>
<td></td>
<td></td>
<td>$O(n^2) \times O(n^2)$ [41]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(WW) $O(n^2) \times O(n)$ [38]</td>
</tr>
<tr>
<td>SEPARATION</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>U</strong></td>
<td>$(l, R, S)$</td>
<td>$Above$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$SepO(n^3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[48], [43]</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>CC</strong></td>
<td>$left/right, submod$</td>
<td>$SEP(O(easy?))$</td>
<td>$hole/bucket$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[9]</td>
<td>$SEP?$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[40]</td>
</tr>
<tr>
<td>OPTIMIZATION</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>U</strong></td>
<td>$O(n \log n)$</td>
<td>$O(n)$</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>[46, 1, 14]</td>
<td>[46, 1, 14]</td>
<td>–</td>
</tr>
<tr>
<td><strong>CC</strong></td>
<td>?</td>
<td>?</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>WW $O(n \log n) v H$</td>
</tr>
</tbody>
</table>

Table 3: Model $PROB - \{CC, U\} - SC$ with Start-Ups
Problem 1. Reformulation

Classification NL=4, NK=1, NT=30, NL=1: WW-CC-SC/SC1

Two possible relaxations for which a tight reformulation is known.

**WW-U-SC and WW-CC**

Reformulation **WW-U-SC**

\[ s_{t-1} \geq \sum_{j=t}^{l} d_j (1 - y_t - z_{t+1} - \ldots - z_j) \]

Reformulation **WW-CC**

\[ s_{k-1} \geq C \sum_{t \in [k, n]} f_t^{k} \delta_t^{k} + C \mu_k \forall k \]

\[
\sum_{u=k}^{t} y_u \geq \sum_{\tau \in \{0\} \cup [k, n]} \left[ \frac{d_{k\tau}}{C} - f_\tau^{k} \right] \delta_\tau^{k} - \mu_k \forall k, t, k \leq t
\]

\[
\sum_{t \in \{0\} \cup [k, n]} \delta_t^{k} = 1 \forall k
\]

\[ \mu_k \geq 0, \delta_t^{k} \geq 0, \text{ for } t \in \{0\} \cup [k, n], \forall k \]

\[ 0 \leq y_t \leq 1 \text{ for } t = 1, \ldots, NT \]

(here \( f_0^{k} = 0 \) and \( f_\tau^{k} = \frac{d_{k\tau}}{C} - \left\lfloor \frac{d_{k\tau}}{C} \right\rfloor \)).
Problem 1. Results of Reformulation

Instance cl-1a consists of the original formulation.
Instance cl-1b is with the addition of the inequalities for $WW - U - SC$.
Instance cl-1c has in addition the reformulation of $WW - CC$ for each item.

<table>
<thead>
<tr>
<th>instance</th>
<th>m</th>
<th>n</th>
<th>int</th>
<th>LP</th>
<th>XLP</th>
<th>IP</th>
<th>secs</th>
<th>nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>cl-1a</td>
<td>511</td>
<td>720</td>
<td>120</td>
<td>1509.1</td>
<td>3549.6</td>
<td>4414.2</td>
<td>5000*</td>
<td>$3.8 \times 10^5$</td>
</tr>
<tr>
<td>cl-1b</td>
<td>2354</td>
<td>720</td>
<td>120</td>
<td>3800.6</td>
<td>4305.1</td>
<td>4404.5</td>
<td>383</td>
<td>3826</td>
</tr>
<tr>
<td>cl-1c</td>
<td>4454</td>
<td>2824</td>
<td>120</td>
<td>4309.9</td>
<td>4310.5</td>
<td>4404.5</td>
<td>82</td>
<td>175</td>
</tr>
</tbody>
</table>

Table 4: Results for Problem 1

* indicates that the run was terminated before optimality was proved. For formulation cl1a the best lower bound on termination was 4251.2 leaving a gap of 3.7%.
Problem 2. Reformulation

Observation 1 Reformulation of changeover variables

Consider the polyhedron representing the flow of a single unit from item to item over time:

\[
\sum_i \chi_{t}^{ij} = y^j_t \ \forall j, t \\
\sum_j \chi_{t}^{ij} = y^i_{t-1} \ \forall i, t \\
\sum_i y^i_0 = 1 \\
\chi_{t}^{ij} \geq 0 \ \forall i, j, t
\]

shown in Figure 1, where \(y^i_t\) is the flow through node \((i, t)\) and \(\chi_{t}^{ij}\) is the flow from node \((i, t - 1)\) to node \((j, t)\). This formulation representing a flow of one unit (or a path) is as tight as possible.
Observation 2: Inclusion of start-up variables.

\[ z^{i^j}_t = \sum_{i:i\neq j} \chi^{i^j}_t \] gives a start-up variable.

Observation 3: Single item subproblems: DLS-CC-SC

Use the reformulation of Van Eyl (WW).

Proposition 1. Suppose that \( d_{t_1} = \ldots = d_{t_p} = 1 \) with \( t < t_1 < \ldots < t_p = l \) and \( d_r = 0 \) in intervening periods. The inequality

\[ s^{i}_{t-1} + \sum_{u=t}^{t+p-1} y^i_u + \sum_{u=t+1}^{t+p-1} (d_{ul} - (t+p-u))z_u + \sum_{u=t+p}^{l} d_{ul}z_u \geq p \]

is valid.
Problem 2: Results

Instance cl2-NTa is the initial formulation
Instance cl2-NTb is the formulation with reformulation from Observations 1 and 2.
Instance cl2-NTc also includes the reformulation of $DLS - CC - SC(WW)$.

<table>
<thead>
<tr>
<th>instance</th>
<th>m</th>
<th>n</th>
<th>int</th>
<th>LP</th>
<th>XLP</th>
<th>IP</th>
<th>secs</th>
<th>nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>cl2-35a</td>
<td>3797</td>
<td>4110</td>
<td>350</td>
<td>27.2</td>
<td>34.7</td>
<td>2056</td>
<td>1800*</td>
<td>51500*</td>
</tr>
<tr>
<td>cl2-35b</td>
<td>2062</td>
<td>5130</td>
<td>690</td>
<td>180.9</td>
<td>531.6</td>
<td>1599</td>
<td>1800*</td>
<td>8000*</td>
</tr>
<tr>
<td>cl2-35c</td>
<td>2599</td>
<td>5130</td>
<td>690</td>
<td>1361.5</td>
<td>1361.5</td>
<td>1387</td>
<td>9</td>
<td>17</td>
</tr>
<tr>
<td>cl2-60c</td>
<td>4817</td>
<td>8880</td>
<td>1190</td>
<td>1453.6</td>
<td>1454.0</td>
<td>1560</td>
<td>17579</td>
<td>8117</td>
</tr>
</tbody>
</table>

Table 5: Results for Problem 2

Note that cl2-35a and cl2-35b are unsolved after 1800 seconds. The best lower bounds obtained are 240.9 and 804.3 respectively.
Problem 3: Reformulation

Classification: Ni=30, NK=1, NT=60, NL=1, DLS-CC-B

Reformulation: Suffices to observe that the flow conservation can be written as

\[ s_t^i \geq C^i \sum_{u=1}^{t} y_u^i - d_{1t}^i \text{ for all } i, t, \]

and then adding the MIR inequalities

\[ s_t^i \geq C^i (1 - f_t^i) \left( \sum_{u=1}^{t} y_u^i - \left\lfloor \frac{d_{1t}^i}{C^i} \right\rfloor \right) \text{ for } i = 1, \ldots, NI, t = 1, \ldots, NT \]

where \( f_t^i = \frac{d_{1t}^i}{C^i} - \left\lfloor \frac{d_{1t}^i}{C^i} \right\rfloor \) gives the convex hull even with \( NI > 1. \)
Problem 3: Results

<table>
<thead>
<tr>
<th>Instance</th>
<th>#col</th>
<th>#row</th>
<th>#non-0</th>
<th>LP</th>
<th>LP time</th>
<th>Best LB</th>
<th>Best UB</th>
<th>IP time</th>
</tr>
</thead>
<tbody>
<tr>
<td>DLSB-p</td>
<td>5160</td>
<td>1621</td>
<td>12668</td>
<td>1627056</td>
<td>1</td>
<td>1759969</td>
<td>2188995</td>
<td>900*</td>
</tr>
<tr>
<td>DLSB-o</td>
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<td>16723</td>
<td>2507612</td>
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<td>2711504</td>
<td>3274036</td>
<td>900*</td>
</tr>
<tr>
<td>DLSB-pr</td>
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<td>51974</td>
<td>1858181</td>
<td>7</td>
<td>1858181</td>
<td>1858181</td>
<td>7</td>
</tr>
<tr>
<td>DLSB-or</td>
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<td>3913</td>
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<td>3069539</td>
<td>3069539</td>
<td>56</td>
</tr>
</tbody>
</table>

Table 6: $DLSB$ instances before and after reformulation
Problem 4: Wagner-Whitin with Safety Stocks

\[ s_{t-1} + x_t = d_t + s_t \ \forall t \]

\[ x_t \leq Cy_t \ \forall t \]

\[ \sigma_t \geq s_t - SS \ \forall t \]

\[ x_t, s_t \geq 0, y_t \in \{0, 1\} \ \forall t \]

\[ H(s) \]

Again mixing suffices.
Problem 4: Initial Stocks/Lower Bounds

\[ s_{k-1} \geq S > 0. \] In place of the inequality

\[ s_{k-1} + \sum_u d_{ul}y_u \geq d_{kl}, \]

use the values \( d_{ku} - S \) to calculate new demands \( \tilde{d}_u \) for \( u = k, \ldots, n \), and use this to calculate new demands \( \tilde{d}_{ul} \) for \( u = k, \ldots, l \).

Example.

\((d_2, d_3, d_4) = (2, 3, 6)\) and \( s_1 \geq 4 \).

Basic Wagner-Whitin inequality

\[ s_1 + 11y_2 + 9y_3 + 6y_4 \geq 11. \]

Using the lower bound with \( s' = s_1 - 4 \), \((d_2 - 4, d_{23} - 4, d_{24} - 4)^+ = (0, 1, 7)\).

So \((\tilde{d}_2, \tilde{d}_3, \tilde{d}_4) = (0, 1, 6)\).

The new inequality is

\[ s_1 + 7y_2 + 7y_3 + 6y_4 \geq 7 + 4, \text{ or } s_1 + 7y_2 + 7y_3 + 6y_4 \geq 11. \]
Problem 4: Results

<table>
<thead>
<tr>
<th>Instance</th>
<th>#rows</th>
<th>#cols</th>
<th>#int</th>
<th>LP</th>
<th>LP time</th>
<th>Best LB</th>
<th>Best UB</th>
<th>IP time</th>
</tr>
</thead>
<tbody>
<tr>
<td>worms1</td>
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<td>15685</td>
<td>3586</td>
<td>5572357</td>
<td>5585163</td>
<td>5586203</td>
<td>5587910</td>
<td>900*</td>
</tr>
<tr>
<td>w1d</td>
<td>32975</td>
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<td>5587453</td>
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<td>5587910</td>
<td>435</td>
</tr>
</tbody>
</table>

Table 7: *worms* instances before and after reformulation
Mixed Models: WW-U-{B,SC}

Consider the problem

\[
\begin{align*}
\min & \sum_t h_t s_t + \sum_t b_t r_t + \sum_t f_t y_t + \sum_t g_t z_t \\
& s_{t-1} - r_{t-1} + x_t = d_t + s_t - r_t \quad \forall \ t \\
& x_t \leq M y_t \quad \forall \ t \\
& z_t - w_{t-1} = y_t - y_{t-1} \quad \forall \ t \\
& z_t \leq y_t \quad \forall \ t \\
& x, s, r \in \mathbb{R}_+^n, y, z \in \{0, 1\}^n
\end{align*}
\]

The extended formulation \(Q_{WW-U-B-SC}^{\infty}\):

\[
\begin{align*}
\alpha_t + y_t + \beta_t &= 1 \quad \forall \ t \text{ with } d_t > 0 \\
\beta_{t+1} + z_{t+1} &\geq \beta_t \quad \forall \ t \\
\sum_{u=t}^{k} d_u (\alpha_u - \sum_{j=t}^{u-1} w_j) &\geq s_{t-1} \quad \forall \ t, k \text{ with } t \leq k \\
r_t &\geq \sum_{u=k}^{t} d_u (\beta_u - \sum_{j=u+1}^{t} z_j) \quad \forall \ k, t \text{ with } k \leq t \\
& z_t - w_{t-1} = y_t - y_{t-1} \quad \forall \ t \\
& z_t \leq y_t \quad \forall \ t \\
x, s, r \in \mathbb{R}_+^n, y, z \in [0, 1]^n
\end{align*}
\]
LS-U-\{B,Sales,Piecewise Concave Production Costs\}

The problem can be formulated as

\[
\begin{align*}
& \text{min } \sum_{k,t} p_t^k x_t^k + \sum_{k,t} f_t^k y_t^k + \sum_{k,t} h_t^k s_t^k + \sum_{k,t} b_t^k r_t^k - \sum_{l,t} e_t^l v_t^l \\
& \quad s_{t-1} - r_{t-1} + \sum_k x_t^k = d_t + \sum_l v_t^l + s_t - r_t \forall t \\
& \quad v_t^l \leq V_t^l \forall l, t \\
& \quad x_t^k \leq M y_t^k \forall k, t \\
& \quad s, r, x, v \geq 0, \ y \in \{0, 1\}^n
\end{align*}
\]

where we assume that \(0 \leq f_t^1 \leq f_t^2 \leq \ldots\) for all \(t\), and \(e_t^1 \leq e_t^2 \ldots\) for all \(t\).

Note that if \(f_t^k = 0\), \(x_t^k\) can be viewed as the amount bought in from outside in period \(t\).
An extended formulation

Letting $\bar{V}_l^t = \sum_{\lambda} V_{t}^{\lambda}$ for all $l, t$, we observe that, because $e_{t}^{1} \leq e_{t}^{2} \ldots$ for all $t$, $\sum_{\lambda} v_{t}^{\lambda} = \bar{V}_{t}^{l}$ for some $l$ and all $t$.

New variables in our extended formulation.

$\alpha_{ut}^{kl} = 1$ if production takes place in period $u$ using production type $k$ to satisfy a demand of $d_{t} + \bar{V}_{t}^{l}$ in period $t$.

$z_{ut}^{k} = 1$ if production takes place in period $u$ using production type $k$ to satisfy the demand in $t$. 
The Formulation

$$\min \sum_{k,t} p_t^k x_t^k + \sum_{k,t} f_t^k y_t^k + \sum_{k,t} h_t^k s_t^k + \sum_{k,t} b_t^k r_t^k - \sum_{l,t} e_t^l v_t^l$$

$$z_{ut} = \sum_l \alpha_{ut}^{kl} \text{ for } 1 \leq u, t \leq n, \forall k$$

$$\sum_k \sum_u z_{ut}^k = 1 \text{ for } 1 \leq t \leq n,$$

$$y_t^k \geq z_{tt}^k \text{ for } 1 \leq t \leq n,$$

$$z_{ut}^k \geq z_{u,t+1}^k \text{ for } 1 \leq u \leq t \leq n, \forall k$$

$$z_{ut}^k \geq z_{u,t-1}^k \text{ for } 1 \leq t \leq u \leq n, \forall k$$

$$v_t^l = \sum_k \sum_u \sum_{\lambda: \lambda \geq l} V_t^\lambda \alpha_{ut}^{kl} \text{ for } 1 \leq t \leq n, \forall l$$

$$x_u^k = \sum_l \sum_t (d_t + \tilde{V}_t^l) \alpha_{ut}^{kl} \text{ for } 1 \leq u \leq n, \forall k$$

$$s_{t-1} = \sum_{k,l} \sum_{u,\tau: u < t \leq \tau} (d_\tau + \tilde{V}_\tau^l) \alpha_{u\tau}^{kl} \text{ for } 1 \leq t \leq n$$

$$r_t = \sum_{k,l} \sum_{u,\tau: \tau \leq t < u} (d_\tau + \tilde{V}_\tau^l) \alpha_{u\tau}^{kl} \text{ for } 1 \leq t \leq n$$

$$\alpha_{ut}^{kl} \geq 0, z_{ut}^k \geq 0, 0 \leq y_t \leq 1 \forall t$$
Figure 2: Formulation as a fixed charge network flow
Open Questions

Compact Formulations?

\[ WW - CC - B \text{ recent result } O(n^3) \times O(n^2) \]
\[ WW - CC - ST \]
\[ DLS - CC - \{B, ST\} \]

Approximate Formulations
What to do if \( n = NT = 60/90? \) Even \( O(n^2) \) becomes problematic.

Resource Constraint Submodels

\[ BB - SET - ST \]

How to generate cuts easily?
Extend results of Miller et al.
Supply Chain Models?
References


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