Complete Segal Spaces, Segal Categories, and S-Categories

Problem: Would like Quillen equivalences of model category structures between:

\[
\begin{align*}
S\text{-Categories} & \quad \uparrow \\ & \quad \Downarrow \\
\text{Segal categories} & \quad \uparrow \\
& \quad \Downarrow \\
\text{Complete Segal spaces}
\end{align*}
\]
Complete Segal Spaces

Definition: A bisimplicial set is a functor $X: \Delta^{op} \to \text{SSets}$.

Definition: Let $W$ be a (Reedy fibrant) bisimplicial set. $W$ is a Segal space if for each $k \geq 2$ the Segal map

$$W_k \rightarrow W_1 \times_{W_0} \cdots \times_{W_0} W_1$$

is a weak equivalence of simplicial sets.

Given a Segal space, we can apply "categorical" terms to it.
Let $W$ be a Segal space.

Define $\text{Ob}(W) = W_{0,0}$
(0-set of the simplicial set $W_0$)

If $x, y \in \text{Ob}(W)$, define the mapping space $\text{map}_W(x, y)$ to be the fiber over $(x, y)$ in the map $(d_1, d_0) : W_1 \to W_0 \times W_0$.

(The Reedy fibrant condition guarantees that this definition is homotopy invariant.)

Given two "maps" $f, g \in \text{map}_W(x, y)$, they are homotopic $(f \sim g)$ if they lie in the same component.
There is a notion of "composition" of maps.

A map $g \in \text{map}_w(x,y)$ is a homotopy equivalence if there exist maps $f, h : \text{map}_w(y,x)$ such that $g \circ f \sim \text{id}_x$ and $h \circ g \sim \text{id}_y$.

Proposition: (Rezk)

Any map in the same component as a homotopy equivalence is itself a homotopy equivalence.

Thus, we can define the space of homotopy equivalences $\text{Whoequiv} \subset W_1$
Since identity maps are homotopy equivalences, the map

\[ s_0 : W_0 \to W_1 \]

factors through \( \text{Whoequiv} \):

\[ W_0 \xrightarrow{s_0} W_1 \xrightarrow{\text{inclusion}} W_{\text{whoequiv}} \]

**Definition:** A Segal space \( W \) is a **complete Segal space** if the map

\[ W_0 \to \text{Whoequiv} \]

is a weak equivalence of simplicial sets.

Let \( C \) be a simplicially enriched category. Taking its nerve yields a Segal category. Can "localize" to obtain a complete Segal space.
Example: let $C$ be a (discrete) category. The resulting complete Segal space looks like:

\[
\begin{array}{ccc}
\langle x, y \rangle & \Rightarrow & B \text{Aut}(\text{Hom}(x, y)) \\
\Rightarrow & & \Rightarrow \\
\text{equivalence classes of objects} & \Rightarrow & \langle x \rangle \ B \text{Aut}(x)
\end{array}
\]

Theorem: (Rezk)
There is a model category structure on the category of bisimplicial sets such that:

- the weak equivalences are the maps $f: X \to Y$ such that for any complete Segal space $W$, the map $f^*: \text{Hom}(Y, W) \to \text{Hom}(X, W)$ is a weak equivalence of simplicial sets
- the cofibrations are inclusions
- fibrant objects = complete Segal spaces
Recall: There is a model category structure on the category of Segal precategories such that the fibrant objects are Segal categories.

There is a Quillen equivalence of model categories:

```
"Segal category"  inclusion  "complete Segal space"
```

```
model category  right adjoint  "discretizes"
```

model category
S-Categories - Model Category Structure

Let $C$ be an $S$-category.

Define $\pi_0 C$ to be the category of components of $C$.

A morphism $g \in \text{Hom}_C(x, y)_0$ is a homotopy equivalence if there is a morphism $g' \in \text{Hom}_C(y, x)_0$ such that $g'g$ is in the same component as $\text{id}_x \in \text{Hom}_C(x, x)_0$ (and similarly for $gg'$).
Theorem: There is a model category structure on $S\text{-Cat}$ in which:

- the weak equivalences $f : C \to \mathcal{D}$ satisfy
  - for any $a_1, a_2 \in \text{Ob}(C)$, the map $\text{Hom}_C(a_1, a_2) \to \text{Hom}_\mathcal{D}(fa_1, fa_2)$ is a weak equivalence of simplicial sets, and
  - $\pi_0 f : \pi_0 C \to \pi_0 \mathcal{D}$ is an equivalence of categories
- the fibrations $f : C \to \mathcal{D}$ satisfy
  - for any $a_1, a_2 \in \text{Ob}(C)$, the map $\text{Hom}_C(a_1, a_2) \to \text{Hom}_\mathcal{D}(fa_1, fa_2)$ is a fibration of simplicial sets, and
  - for any $a_1 \in C$, $b \in \mathcal{D}$, and homotopy equivalence $g : fa_1 \to b$, there is a homotopy equivalence $d : a_1 \to a_2$ in $C$ such that $fd = g$.

\[
\begin{array}{c}
C \xrightarrow{a_1 \to a_2} \\
\downarrow \\
\mathcal{D} \xrightarrow{fa_1 \to b}
\end{array}
\]
Hope: There is a Quillen equivalence $\mathcal{S}$-Cat $\iff$ "Segal categories"

Problem: The model category structure on Segal precategories does not work.