Weak Categories

Slides prepared by
Timothy Porter and Peter May
for a talk given by
André Joyal, Timothy Porter, and Peter May
Survey: definitions and comparisons of various notions of ‘weak category’ and of enriched and internal variants.

- Quasi-categories
- Segal cats and complete Segal cats
- $A_\infty$-categories
- Top-Cats and $S$-cats ($S = \text{sSets}$)
- Philosophy: weak $(\infty, 1)$-categories
- An analogue: weak complicial sets
Two related problems:

(i) (‘small’ version) need ‘adequate’ kinds of (small) weak (−) categories, for instance to model homotopy types, stacks, higher dimensional automata, and so forth.

(−) might be $n$ or $\infty$, or $\cdots$

(ii) (‘large’ version) need an adequate weak (−) category theory, including the essence of homotopy coherence to handle/study the objects of (i), and to implement potential applications.
Weakening categories

Understand $\text{Cat}$ in $\mathcal{S}$ (simplicial sets) then weaken structure.

**Quasi-categories**

*If $\mathcal{C}$ is a (small) category, then $\text{Ner}(\mathcal{C})$ is a weak or restricted Kan complex.*

(Fillers for inner horns.)

*If $\mathcal{G}$ is a groupoid, then $\text{Ner}(\mathcal{G})$ is a Kan complex.*

**Definition**

A *quasi-category* is a restricted Kan complex.
Simplicial characterization of categories

Segal Maps

Let $p > 0$. We have increasing maps $e_i : [1] \to [p]$ given by

$e_i(0) = i$ and $e_i(1) = i + 1$.

Note that $e_i(1) = e_{i+1}(0)$.

For a simplicial set $A : \Delta^{op} \to Sets$, evaluate $A$ on the $e_i$ to get functions $A_p \to A_1$.

These yield a cone diagram,

for instance, when $p = 3$: 
We get “Segal maps”

\[\delta[p] : A_p \to A_1 \times A_0 A_1 \times A_0 \ldots \times A_0 A_1.\]
If \( A = Ner(C) \) for a small category \( C \), then the Segal maps are bijections.

The converse holds:

If \( A \) is a simplicial set such that the Segal maps are bijections, then there is a category structure on the directed graph

\[
A_1 \rightarrow A_0
\]

making it into a category whose nerve is isomorphic to \( A \).
Complete Segal categories

Definition

Let $\mathcal{C}$ be a Cartesian monoidal category and $\mathcal{W}$ be a subcategory of “weak equivalences”.

A complete Segal category in $(\mathcal{C}, \mathcal{W})$ is a simplicial object $A$ in $\mathcal{C}$ such that the Segal maps of $A$ are in $\mathcal{W}$.

$(\mathcal{C}, \mathcal{W}) = (\text{Sets, Bijections})$: Get $\text{Cat}$.

Object sets play a distinguished role.
Segal categories

Assume that $\mathcal{C}$ has coproducts.

Define $\delta: \text{Sets} \to \mathcal{C}$ via coproducts of copies of the terminal object $\ast$.

This defines “discrete objects” in $\mathcal{C}$.

Definition

A \textit{Segal category} in $(\mathcal{C}, \mathcal{W})$ is a complete Segal category $\mathcal{A}$ such that $\mathcal{A}_0$ is a discrete object of $\mathcal{C}$. 
The original notion of Segal space is 
\((\mathcal{C}, \mathcal{W}) = (\mathcal{T}op, \text{weak equivalence})\).

Can replace \(\mathcal{T}op\) by \(\mathcal{S}\) or \(\mathcal{C}at\).

The starting point, \(n = 2\), of the Tamsamani-Simpson definition of weak \(n\)-category is 
\((\mathcal{C}, \mathcal{W}) = (\mathcal{C}at, \text{equivalence})\).
$A_\infty$ spaces and categories

Segal categories in $(\mathcal{T}op, \text{weak equiv})$

with object set $\ast$ model loop spaces.

Stasheff associahedra operad $\mathcal{K} = \{\mathcal{K}(q)\}$.

$A_\infty$ spaces,

$$\mathcal{K}(q) \times X^j \to X,$$

also model loop spaces.

Many object version:

$A_\infty$-category in $\mathcal{T}op$. 
DG $A_\infty$ algebras and categories

$C_\star(\mathcal{K})$: operad of chain complexes.

DG $A_\infty$ algebra:

$$C_\star(\mathcal{K}(q)) \otimes A^q \to A.$$  

Many object version:

DG $A_\infty$-category.

(Fukaya, Kontsevich, mirror symmetry.)

A generalization gives a crude initial pattern for the operadic definitions of weak $n$-categories of Batanin, Leinster, Trimble, and May.
$A_\infty$-category

Let $\mathcal{C}$ be a monoidal category with unit object $*$ and a subcategory $\mathcal{W}$. An $A_\infty$-operad in $(\mathcal{C}, \mathcal{W})$ is an operad $\mathcal{O}$ in $\mathcal{C}$ with an augmentation $\mathcal{O}(q) \to \text{Ass}$ such that $\mathcal{O}(q) \to *$ is in $\mathcal{W}$ for each $q$.

Definition

An $A_\infty$-category $\mathcal{C}$ in $(\mathcal{C}, \mathcal{W})$ is an $\mathcal{O}$-category for an $A_\infty$ operad $\mathcal{O}$. 

\[ \mathcal{O}(q) \otimes \mathcal{C}(S_{q-1}, S_q) \otimes \ldots \otimes \mathcal{C}(S_0, S_1) \longrightarrow \mathcal{C}(S_0, S_q) \]
**Enriched $\mathcal{V}$-categories**

Let $\mathcal{V}$ be a closed symmetric monoidal category with unit object $I$.

Change point of view, allow classes of objects in our structured categories $\mathcal{C}$.

Enrichment over $\mathcal{V}$:

Objects $\mathcal{C}(X, Y)$ of $\mathcal{V}$ for objects $X$ and $Y$ of $\mathcal{C}$

\[ \mathcal{C}(S, T) \otimes \mathcal{C}(R, S) \to \mathcal{C}(R, T) \]

\[ \text{Id}: I \to \mathcal{C}(S, S) \]

$\mathcal{V}$-$\text{Cat} = \mathcal{V}$-enriched small categories.
Internal categories in $\mathcal{V}$

Let $\mathcal{V}$ be bicomplete.

An internal category $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{O}(\mathcal{C})$ and $\mathcal{M}(\mathcal{C})$ of $\mathcal{V}$ with maps $S$, $T$, $Id$, and $C$ in $\mathcal{V}$ which satisfy the category axioms.

$\text{Cat}(\mathcal{V}) = \text{internal categories in } \mathcal{V}$.

Functor $\mathcal{V}$-cat $\rightarrow \text{Cat}(\mathcal{V})$

$\delta$ on objects gives $\mathcal{O}(\mathcal{C})$, coproduct of the $\mathcal{C}(S, T)$ gives $\mathcal{M}(\mathcal{C})$.

Get “object-discrete internal categories”.
Why enrich and why $S$?

Problems (i) and (ii) suggest that a ‘good’ category of (small) weak (−) categories might be expected to itself be a (large) weak (−) category and also a category ‘weakly enriched’ over weak (−) categories.

Categories are intrinsically intertwined with simplicial sets, and simplicial sets are good for doing homotopy theory.

$S$-enriched categories may be a good starting point.
Digression on alternatives

Joyal: Enrichment over quasi-categories may be an even better choice.

Porter: I do not favour Top-enriched categories as I have found them awkward to use in the applications that I have been studying.

May: I favor Top-enriched categories as they are indispensable to many applications I have been studying.
Example: topologized fundamental groupoid

Internal categories in Top are good.

\( \Pi X \) has object space \( X \), morphism space of Moore paths (varying length) in \( X \), and \( X \simeq B\Pi X \). Get quotient topological groupoid \( \pi^{Top} X \) (identify homotopic paths) such that

\[
B_{\pi^{Top} X} \simeq B\pi X.
\]

(\( \pi X \) is the discrete fundamental groupoid).

\[
X \simeq B\Pi X \to B\pi^{Top} X \simeq B\pi X.
\]
How to enrich over $\mathcal{S}$

Let $\mathcal{C}$ be symmetric monoidal and $F : \Delta \rightarrow \mathcal{C}$ be a cosimplicial object.

$$[n] \rightarrow \mathcal{C}(X \otimes F([n]), Y)$$
defines enriched Hom.

Top, $\mathcal{S} Ch_k$ (chain complexes).

$\mathcal{S} Sm/k$ (Smooth schemes over $k$).

$\mathcal{C} at$: two interesting choices of $F$.

$[n] \mapsto$ Total order with $n + 1$ objects,

$[n] \mapsto$ Groupoid with $n + 1$ uniquely isomorphic objects.

Analogous construction works when $\mathcal{C}$ is tensored over $\mathcal{S}$. 
In general:

For any category $\mathcal{C}$, $\text{Simp}(\mathcal{C}) = \mathcal{C}^{\Delta^{op}}$ is an $\mathcal{S}$-category.

Let $K$ be a simplicial set. The comma category $\Delta/K$ has objects $([n], x)$ with $x \in K_n$. The morphisms

$$\mu : ([n], x) \rightarrow ([m], y)$$

are those $\mu : [n] \rightarrow [m]$ in $\Delta$ such that $K(\mu)(y) = x$.

There is a forgetful functor

$$\delta_K : \Delta/K \rightarrow \Delta, \quad \delta_K([n], x) = [n].$$

Given $X, Y \in \text{Simp}(\mathcal{A})$, define

$$\text{Simp}(\mathcal{A})(X, Y)_n = \text{Nat}(X\delta_{\Delta[n]}^{op}, Y\delta_{\Delta[n]}^{op}).$$
**S-categories versus $\text{Simp}(\text{Cat})$**

Let $\mathbb{B} : \Delta^{op} \to \text{Cat}$ be a simplicial object in $\text{Cat}$. It has a simplicial set $B$ of objects. Say that $\mathbb{B}$ is object discrete if $B$ is discrete. Then, for $(x, y) \in B$, let

$$\mathbb{B}(x, y)_n = \{ \sigma \in \mathbb{B}_n | S(\sigma) = x, T(\sigma) = y \}.$$  

With faces and degeneracies induced from those of $\mathbb{B}$, $\{ \mathbb{B}(x, y)_n | n \in \mathbb{N} \}$ is a simplicial set $\mathbb{C}(x, y)$. Levelwise composition in $\mathbb{B}$ induces

$$\mathbb{C}(x, y) \times \mathbb{C}(y, z) \to \mathbb{C}(x, z).$$

This gives an $\mathcal{S}$-enriched category $\mathbb{C}$. 
Conversely, a small $S$-category $\mathcal{C}$ gives an object discrete simplicial category $\mathbb{B}$: $S$-categories are object discrete simplicial categories.

Recall: Segal categories are object discrete complete Segal categories. When regarding simplicial categories and complete Segal categories as bisimplicial sets, the two notions of “object discrete” differ by which 0th simplicial coordinate is required to be discrete.
Weak categories

Slogan:

“Weak category $= (\infty, 1)$-category”.

“An $(\infty, n)$-category is an $\infty$-category with invertible $i$-morphisms, $i > n$”.

An $(\infty, 0)$-category is a space.
That is, we can take any model for the homotopy theory of spaces, any $(\mathcal{C}, \mathcal{W})$ with correct $\mathcal{C}[\mathcal{W}^{-1}]$.

Examples: $\mathcal{T}op$, $\mathcal{S}$, $\mathcal{C}at$ (Thomason).
Baby Comparison Program

Compare notions of weak category.

Examples ($\mathcal{W} = \text{‘weak equivalence’}$):

- Quasi-category
- Complete Segal category
- Segal category
- $A_\infty$-category
- $S$-category
- $\mathcal{T}op$-category
Weakening of strict $\infty$-Categories

“Complicial set”: a simplicial set $A$ with a stratification by “thin” simplices $t_nA_n \subset A_n$ for all $n$ (including all degenerate simplices) satisfying certain axioms.

Equivalent to strict $\infty$-Categories.

Weak complicial sets: weaken axioms.

Street’s weak $\infty$ categories:

Intrinsic stratification on simplicial sets.

Require: this is a weak complicial set.
S-categories and homotopy coherence

Part of large version of the general problem. Given a small category $\mathbb{D}$, use free-forget monad to get a free simplicial resolution

$$S(\mathbb{D}) \in \text{Simp}(\text{Cat}),$$

similar to free group simplicial resolution or bar resolution. If the monad ‘remembers identities’, then

$$S(\mathbb{D}) \in \mathcal{S}\text{-Cat}.$$

Vogt 1973, Cordier 1982:

A homotopy coherent diagram in $\text{Top}$ of shape $\mathbb{D}$ is the same as an $S$-functor

$$S(\mathbb{D}) \to \text{Top}.$$
Definition (Cordier (1980), based on earlier ideas of Vogt, Boardman-Vogt.)

Let $\mathcal{B}$ be simplicially enriched.

The *homotopy coherent nerve* of $\mathcal{B}$, denoted $\text{Ner}_{h.c.}(\mathcal{B})$, is the simplicial ‘set’ with

$$\text{Ner}_{h.c.}(\mathcal{B})_n = S\text{-Cat}(S[n], \mathcal{B}).$$

If $\mathcal{B}$ is a locally Kan $S$-category, then $\text{Ner}_{h.c.}(\mathcal{B})$ is a quasi-category.

(Cordier–Porter, 1986)

Question: If $\mathcal{B}$ is locally complicial, is $\text{Ner}_{h.c.}(\mathcal{B})$ a weak complicial set?
How else can one construct $S$-cats? Hammocks!

For a category $C$ and subcategory $W$, construct an $S$-category $L^H(C, W)$, or $L^H C$, the *hammock localisation* of $(C, W)$, as follows:

The objects of $L^H C$ are those of $C$.

For objects $X$ and $Y$, the $k$-simplices of $L^H C(X, Y)$ are the “reduced hammocks of width $k$ and any length” between $X$ and $Y$. Such a thing is a commutative diagram of the form
in which
(i) the length $n$ of the hammock is an integer $\geq 0$,

(ii) all the vertical maps are in $W$,

(iii) in each column of horizontal maps, all maps go in the same direction; if they go left, they must be in $W$.

The hammocks are subject to two irredundancy conditions.

(iv) the maps in adjacent columns go in different directions, and

(v) no column contains only identity maps.
Why the hammock localization?

Its category of components is $\mathcal{C}[\mathcal{W}^{-1}]$. Version of inverting $\mathcal{W}$ that retains higher homotopical information. (Massey products or Toda brackets).

Baby comparison should give that the hammock localizations of all models for weak categories have equivalent hammock localizations.

Model category theory shows how.