We introduce a family of generalized Jacobi polynomials (GJPs) with negative integer indexes, which turns out to be the natural basis functions for spectral approximations to PDEs in various situations. As examples of applications, we apply generalized Jacobi spectral methods to general high-order PDEs; Helmholtz equation, and time discretizations.

Spectral methods using GJPs lead to stable well-conditioned algorithms, and more precise error estimates.
Generalized Jacobi polynomials

Let \( \{J_n^{a,b}(x)\}(a, b > -1) \) be the classical Jacobi polynomials orthogonal with 
\( \omega^{a,b}(x) = (1 - x)^a(1 + x)^b \). Define the GJPs by

\[
J_n^{-k,-l}(x) := (1 - x)^k(1 + x)^l J_{n-n_0}^{k,l}(x), \quad n_0 = k + l, \quad k, l \geq 1, \quad k, l \in \mathbb{N}.
\]

GJPs are (i) orthogonal with \((1 - x)^{-k}(1 + x)^{-l}\); (ii) compact combinations of Legendre polynomials; (iii) natural basis functions for PDEs with BCs:

\[
\partial^i_x u(1) = a_i, \quad i = 0, 1, \ldots, k - 1; \quad \partial^j_x u(-1) = b_j, \quad j = 0, 1, \ldots, l - 1.
\]

Consider \( \pi_N^{-k,-l} : L^2_{\omega_{-k,-l}} \to Q_N^{-k,-l} := \text{span}\{J_{k+l}^{-k,-l}, J_{k+l+1}^{-k,-l}, \ldots, J_N^{-k,-l}\} \).

**Theorem 1.** For \( k, l \in \mathbb{N}, \) and \( k, l \geq 1, \)

\[
\| \partial^m_x (u - \pi_N^{-k,-l}u) \|_{\omega_{m-k,m-l}} \leq cN^{m-r} \| \partial^r_x u \|_{\omega_{r-k,r-l}}, \quad 0 \leq m \leq r.
\]
Application to even-order DEs

Consider

\[ u^{(2m)}(x) + \sum_{k=0}^{2m-1} b_{2m-k}(x) u^{(k)}(x) = f(x), \quad \text{in } (-1, 1), \]

\[ u^{(k)}(\pm 1) = 0, \quad 0 \leq k \leq m - 1. \]

It is reduced from semi-implicit time discretization of even-order nonlinear PDEs. The spectral Galerkin approximation is to find \( u_N \in Q^{-m,-m}_N \) s. t.

\[
(\partial_x^m u_N, \partial_x^m v) + (-1)^m (\partial_x^{m-1} u_N, \partial_x^m (b_1 v)) + (-1)^{m-1} (\partial_x^{m-1} u_N, \partial_x^{m-1} (b_2 v)) + \cdots + (b_{2m} u_N, v) = (f, v), \quad \forall v \in Q^{-m,-m}_N.
\]

Under the GJP basis, the matrix corresponding to the leading linear operator \( \partial_x^{2m} \) is identity. Hence, the system is well-conditioned (with condition number independent of \( N \)). Moreover, it leads to optimal error estimates.
Application to odd-order DEs

Consider

\[ (-1)^{m+1}u^{(2m+1)} + (-1)^m \beta u^{(2m-1)} + \alpha u = f, \quad x \in (-1, 1), \quad m \geq 1, \]
\[ u^{(k)}(\pm 1) = u^{(m)}(1) = 0, \quad 0 \leq k \leq m - 1. \]

A semi-implicit time discretization of KdV-type equations leads to this equation.

**Dual Petrov-Galerkin formulation:** Find \( u_N \in Q_N^{-(m+1),-m} \) such that

\[ -(\partial_x^m u_N, \partial_x^{m+1} v) - \beta (\partial_x^{m-1} u_N, \partial_x^m v) + \alpha (u_N, v) = (f, v), \quad \forall v \in Q_N^{-m,-(m+1)}. \]

**Well-posedness:** For \( u_N \in Q_N^{-(m+1),-m} \),

\[ -(\partial_x^{m+1} u_N, \partial_x^m (u_N \omega^{-1,1})) = (2m + 1) \int_{-1}^{1} \left( \partial_x^m \left( \frac{u_N}{1-x} \right) \right)^2 dx. \]
Under the GJP basis, the matrix of the leading operator is an identity matrix, the system is sparse and well-conditioned. Consider the fifth-order equation:

\[ u^{(5)} + a_1(x)u' + a_0(x)u = f, \quad \text{in } (-1, 1), \quad u(\pm1) = u'(\pm1) = u''(1) = 0. \]

The condition numbers of the collocation method (COL) and the new method (GJS) are listed below:

<table>
<thead>
<tr>
<th>( N )</th>
<th>Method</th>
<th>( a_0 = 0 )</th>
<th>( a_0 = 10 )</th>
<th>( a_0 = 50 )</th>
<th>( a_0 = 100x )</th>
<th>( a_0 = 10e^{10x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>COL</td>
<td>3.30E+05</td>
<td>3.77E+05</td>
<td>4.46E+05</td>
<td>2.49E+05</td>
<td>4.09E+05</td>
</tr>
<tr>
<td></td>
<td>GJS</td>
<td>1.00</td>
<td>1.07</td>
<td>1.42</td>
<td>1.62</td>
<td>33.05</td>
</tr>
<tr>
<td>32</td>
<td>COL</td>
<td>2.70E+08</td>
<td>2.78E+08</td>
<td>3.36E+08</td>
<td>1.37E+08</td>
<td>8.22E+08</td>
</tr>
<tr>
<td></td>
<td>GJS</td>
<td>1.00</td>
<td>1.07</td>
<td>1.42</td>
<td>1.62</td>
<td>33.05</td>
</tr>
<tr>
<td>64</td>
<td>COL</td>
<td>2.58E+11</td>
<td>2.64E+11</td>
<td>4.43E+11</td>
<td>8.11E+10</td>
<td>1.37E+11</td>
</tr>
<tr>
<td></td>
<td>GJS</td>
<td>1.00</td>
<td>1.07</td>
<td>1.42</td>
<td>1.62</td>
<td>33.05</td>
</tr>
<tr>
<td>128</td>
<td>COL</td>
<td>2.05E+14</td>
<td>2.10E+14</td>
<td>2.39E+14</td>
<td>1.86E+14</td>
<td>2.64E+14</td>
</tr>
<tr>
<td></td>
<td>GJS</td>
<td>1.00</td>
<td>1.07</td>
<td>1.42</td>
<td>1.62</td>
<td>33.05</td>
</tr>
</tbody>
</table>
Application to Helmholtz equation

Consider the Helmholtz equation in a disk/disk layer or sphere/ spherical shell:

\[- \Delta U - k^2 U = F, \quad k \gg 1.\]

It suffices to consider the reduced problem in radial direction:

\[- \frac{1}{r^{n-1}} \frac{d}{dr} \left( r^{n-1} \frac{du}{dr} \right) + d_m \frac{u}{r^2} - k^2 u = f, \quad r \in (0, 1),

u'(1) - iku(1) = g, \quad u(0) = 0, \quad \text{if necessary},

\]

where \( d_m = 0, m^2, m(m + 1) \), for \( n = 1, 2, 3 \). Let \( \omega^a(r) = r^a \).

**Weak formulation:** Find \( u \in X \) such that

\[
\mathcal{B}(u, v) := (u', v')_{\omega^{n-1}} + d_m(u, v)_{\omega^{n-3}} - k^2(u, v)_{\omega^{n-1}} - iku(1)v(1)
\]

\[
= (f, v)_{\omega^{n-1}} + gv(1), \quad \forall v \in X, \quad n = 1, 2, 3.
\]
Spectral scheme: Find $u_N \in X_N := X \cap P_N$ such that

$$\mathcal{B}(u_N, v_N) = (f, v_N)_{\omega^{n-1}} + g v_N(1), \quad \forall v_N \in X_N.$$ 

A priori estimates: If $f \in L^2_{\omega^{n-1}}$, then for $w = u, u_N$,

$$\|w\|_{\omega^{n-1}} \leq \frac{c}{k}(|g| + \|f\|_{\omega^{n-1}});$$

$$\|w'\|_{\omega^{n-1}} + \sqrt{d_m}\|w\|_{\omega^{n-3}} \leq c(1 + k^{-1})(|g| + \|f\|_{\omega^{n-1}}).$$

Convergence:

$$|u - u_N|_{1,\omega^{n-1}} + k\|u - u_N\|_{\omega^{n-1}} \leq c(1 + k^2 N^{-1})N^{1-s}\|\partial^s u\|.$$ 

Remarks: Taking different test functions $v = w, rw'$ allows us to derive a prior estimates without using Green’s function. Generalized Jacobi approximations are used to derive optimal error estimates. Using Fourier and spherical harmonic transform, we can extend the 1D results to 2D and 3D.
Application to time discretization

Consider

\[ u_t + \mathcal{L}u + \mathcal{N}u = f, \quad t \in I_t := (0, T), \quad u(x, 0) = u_0(x), \]

where \( \mathcal{L} \) is a leading linear operator and \( \mathcal{N} \) a nonlinear operator. Let \( u_0 = 0 \).

Spectral method in time: Find \( u_N \in V_N := \{ u \in P_N : u(0) = 0 \} \) s.t.

\[
(\partial_t u_N, v)_{I_t} + (\mathcal{L}u, v)_{I_t} + (\mathcal{N}u, v)_{I_t} = (f, v)_{I_t}, \quad \forall v \in V_N^*,
\]

where \( V_N^* := \{ u \in P_N : u(T) = 0 \} \).

Remarks: We can show this scheme is wellposed and enjoys the spectral accuracy in time. The fully discrete scheme coupled with spectral discretization in space is unconditionally stable and suitable for long time simulations. For nonlinear PDEs, Newton-type iterative methods are needed to solve the resulting nonlinear systems.
Numerical results:

Left: Errors at $t = 1, 50, 100$ fifth-order KdV with soliton solution computed by Crank-Nicolson Dual-Petrov-Galerkin method in $[-50, 50]$ with $N = 120$ and $\tau = 0.001$.

Middle: Errors of Helmholtz equation with smooth solution and $k = 100$.

Right: Solution profiles of Burgers equation in $(-1, 1)$ with viscosity $\mu = 0.02$ obtained by spectral scheme in time-space with $N = 16$ (in time) and $M = 100$ (in space).